From Bernstein polynomials to Bernstein copulas

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Dedicated to the 65th birthday of Professor Heiner Gonska

Abstract

In this paper we review Bernstein and checkerboard copulas for arbitrary dimensions and general grid resolutions in connection with discrete random vectors possessing uniform margins, and point out the relation to tensor product Bernstein operators. We further suggest a pragmatic and effective way to fit the dependence structure of multivariate data to Bernstein copulas via rook copulas, a subclass of checkerboard copulas, which is based on the multivariate empirical distribution.

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1 Introduction

In the history of approximation theory, univariate and multivariate Bernstein polynomials have played a central role since the beginning of the 20th century, see, e.g., [11] for a survey of Bernstein polynomials in one variable and [1], chapters 8.4 and 18, for a short treatment of Bernstein polynomials in several variables. They have not only been used to provide a constructive proof of the famous Weierstraß approximation theorem for continuous functions on compact intervals, including explicit estimates for the rate of convergence, but also for more advanced applications in functional analysis and computer aided design, such as Bézier curves and surfaces, see, e.g., [7], [15] and [16]. Here, shape preserving and local smoothness properties of Bernstein polynomials are of central interest, in particular w.r.t. engineering applications. (It might be interesting to note here that Donald Knuth has used Bézier curves for the design of T_EX-fonts.) Applications of Bernstein polynomials for modelling stochastic dependence via so-called copulas have, in contrast, been considered much later.

The use of copulas for modelling and simulation purposes, for instance in risk management, is of increasing importance, see, e.g., [3], section 5.3, or [9], chapter 5, and the references given there. Let us recall that a (d-dimensional) copula C is the cumulative distribution function (cdf) of a random vector $\mathbf{U}=(U_1,\cdots,U_d)$ whose one-dimensional marginal distributions are uniform on the interval $[0,1]$. The following well-known theorem (see, e.g., [9], p. 186) deals with a key property of copulas.

Theorem of Sklar. Let F be the cdf of some random vector $\mathbf{X}=(X_1,\cdots,X_d)$, i.e., $F(x_1,\cdots,x_d)=P(X_1\le x_1,\cdots,X_d\le x_d)$ with marginal cdfs $F_1,\cdots,F_d$. Then there exists a copula $C:[0,1]^d\to[0,1]$ such that $F(x_1,\cdots,x_d)=C(F_1(x_1),\cdots,F_d(x_d))$ for all $x_1,\cdots,x_d\in\mathbb{R}$. If $F_1,\cdots,F_d$ are continuous, then C is uniquely determined.
Vice versa: For a copula C and univariate cdfs $F_1,\cdots,F_d$ the assignment $F(x_1,\cdots,x_d):=C(F_1(x_1),\cdots,F_d(x_d))$ defines the cdf F of some d-variate random vector with marginal cdfs $F_1,\cdots,F_d$.

Thus, the theorem of Sklar states that the cdf F of any d-variate random vector can be written in terms of its marginal distribution functions $F_1,\cdots,F_d$ and a suitable copula C which thus describes the dependence structure of the vector components. Such a decomposition is often very useful in practice; for an exemplary application in the context of Bernstein copulas see Example 4.2.The definition of this specific copula type, constructed by means of Bernstein polynomials, is given in section 2.

The discussion of potential copula models has so far mostly focussed on other types, i.e., either the elliptical case (e.g., the Gaussian and t-copula) or the Archimedean case (e.g., Gumbel-, Clayton-, and Frank-copulas). It seems that the true impact of Bernstein polynomials on copula models has been discovered only more recently, first in the framework of approximation theory (see, e.g., [8], [10], [11]) and later in particular in connection with applications in finance (see, e.g., [2], [5], [6], [13], [14]). Bernstein copulas possess several benefits compared to the traditional approaches:

- Bernstein copulas allow for a very flexible, non-parametric and essentially non-symmetric description of dependence structures also in higher dimensions
- Bernstein copulas approximate any other given copula arbitrarily well
- Bernstein copula densities are given in an explicit form and can hence be easily used for Monte Carlo simulation studies.

In this paper, we review the construction of Bernstein copulas through discrete random vectors with uniform margins (called discrete skeletons), and point out their connection to checkerboard copulas, as discussed, e.g., in [8], [10] and [11], and to Bernstein tensor product operators (cf. the proof of Theorem 2.2). The explicit representation of Bernstein copulas in terms of tensor product Bernstein operators with a discrete skeleton has, to our knowledge, not been stated in the related literature before. This approach, amongst others, opens a pragmatic and storage saving approach to fit the dependence structure of observed data to Bernstein copulas via rook copulas, a special subcase of checkerboard copulas based on the multivariate empirical distribution. The tensor product representation might also be helpful in further studies on global smoothness preservation for copula approximation since it allows a direct transfer of results from multivariate approximation theory (as formulated, e.g., in [4] and [12]) into the copula context.

2 Some simple mathematical facts on Bernstein polynomials and Bernstein copulas

The assertions of the following lemma are well-known in the literature, but for convenience and better understanding in the copula context we give a short proof.

Lemma 2.1. Let $B(m,k,z)=\binom{m}{k}z^k(1-z)^{m-k},\ 0\le z\le 1,\ k=0,\cdots,m\in\mathbb{N}.$ Then we have

$$\int_0^1 m\,B(m-1,k,z)\,dz=1 \ \text{ for } k=0,\cdots,m-1.$$

Further,

$$\frac{d}{dz}B(m,k,z)=m\big[B(m-1,k-1,z)-B(m-1,k,z)\big] \ \text{ for } k=0,\cdots,m$$

with the convention $B(m-1,-1,z)=B(m-1,m,z)=0$. For the Bernstein operator $\mathcal{B}_m$ defined by $\mathcal{B}_m f: z\mapsto\sum_{k=0}^{m} f\left(\frac{k}{m}\right)B(m,k,z)$ for real-valued functions f on $[0,1]$ and $z\in[0,1]$, this yields

$$\frac{d}{dz}\mathcal{B}_m f(z)=m\sum_{k=0}^{m-1}\Delta_m f\left(\frac{k}{m}\right)B(m-1,k,z)$$

where $\Delta_m f(z):=f\left(z+\frac{1}{m}\right)-f(z)$ for $z\in[0,1]$ denotes the forward difference operator.

Proof:

Let $\mathrm{B}(x,y):=\frac{\Gamma(x)\cdot\Gamma(y)}{\Gamma(x+y)}$ for $x,y>0$ denote the Beta function and Γ the Gamma function, as usual. Then

$$\int_0^1 m\,B(m-1,k,z)\,dz=m\binom{m-1}{k}\mathrm{B}(k+1,m-k)=m\binom{m-1}{k}\frac{\Gamma(k+1)\Gamma(m-k)}{\Gamma(m+1)}$$
$$=\frac{m(m-1)!}{k!(m-k-1)!}\times\frac{k!(m-k-1)!}{m!}=1.$$

Further, for $0<k<m$,

$$\frac{d}{dz}B(m,k,z)=k\binom{m}{k}z^{k-1}(1-z)^{m-k}-(m-k)\binom{m}{k}z^k(1-z)^{m-k-1}$$
$$=m\binom{m-1}{k-1}z^{k-1}(1-z)^{(m-1)-(k-1)}-m\binom{m-1}{k}z^k(1-z)^{m-1-k}$$
$$=m\big[B(m-1,k-1,z)-B(m-1,k,z)\big]$$

which, by the above convention, also holds for $k\in\{0,m\}$. The remaining statement follows easily from this. ◆

Theorem 2.1 and Definition. For $d \in \mathbb{N}$ let $\mathbf{U} = (U_1, \cdots, U_d)$ be a random vector whose marginal component U_i follows a discrete uniform distribution over $T_i := \{0, 1, \cdots, m_i - 1\}$ with $m_i \in \mathbb{N}$, $i = 1, \cdots, d$. Let further

$$p(k_1, \cdots, k_d) := P\left(\bigcap_{i=1}^{d} \{U_i = k_i\}\right) \text{ for all } (k_1, \cdots, k_d) \in \bigtimes_{i=1}^{d} T_i.$$

Then

$$c_B^{\mathbf{U}}(u_1, \cdots, u_d) := \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} p(k_1, \cdots, k_d) \prod_{i=1}^{d} m_i B(m_i - 1, k_i, u_i), \ (u_1, \cdots, u_d) \in [0,1]^d$$

defines the density of a d-dimensional copula $C_B^{\mathbf{U}}$, called *Bernstein copula*. We call $c_B^{\mathbf{U}}$ the Bernstein copula density induced by **U**. The vector **U** is also called the discrete skeleton of the Bernstein copula.

Proof. For fixed $1 \le j \le d$ we obtain, according to Lemma 2.1 above,

$$\begin{aligned}
\int_0^1 c_B^{\mathbf{U}}(u_1, \cdots, u_d)\, du_j &= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} p(k_1, \cdots, k_d) \prod_{\substack{i=1 \\ i \ne j}}^{d} m_i B(m_i - 1, k_i, u_i) \int_0^1 m_j B(m_j - 1, k_j, u_j)\, du_j \\
&= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_d=0}^{m_d-1} p(k_1, \cdots, k_d) \prod_{\substack{i=1 \\ i \ne j}}^{d} m_i B(m_i - 1, k_i, u_i) \\
&= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_{j-1}=0}^{m_{j-1}-1} \sum_{k_{j+1}=0}^{m_{j+1}-1} \cdots \sum_{k_d=0}^{m_d-1} \left(\sum_{k_j=0}^{m_j-1} p(k_1, \cdots, k_d) \right) \prod_{\substack{i=1 \\ i \ne j}}^{d} m_i B(m_i - 1, k_i, u_i) \\
&= \sum_{k_1=0}^{m_1-1} \cdots \sum_{k_{j-1}=0}^{m_{j-1}-1} \sum_{k_{j+1}=0}^{m_{j+1}-1} \cdots \sum_{k_d=0}^{m_d-1} P\left(\bigcap_{\substack{i=1 \\ i \ne j}}^{d} \{U_i = k_i\} \right) \prod_{\substack{i=1 \\ i \ne j}}^{d} m_i B(m_i - 1, k_i, u_i) \\
&= c_B^{\mathbf{U}^{\setminus j}}(u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_d)
\end{aligned}$$

for $(u_1, \cdots, u_{j-1}, u_{j+1}, \cdots, u_d) \in [0,1]^{d-1}$, where $\mathbf{U}^{\setminus j} = (U_1, \cdots, U_{j-1}, U_{j+1}, \cdots, U_d)$ (note that for $j = 1$, the symbol $\mathbf{U}^{\setminus j}$ reads $(U_2, \cdots, U_d)$, likewise for $j = d$). We thus obtain another Bernstein copula density, but of dimension $d - 1$ instead of d. Continuing integration according to the remaining variables except for the variable u_r for fixed $1 \le r \le d$, we end up with

$$\begin{aligned}
\int_0^1 \cdots \int_0^1 c(u_1, \cdots, u_d)\, du_1 \cdots du_{r-1}\, du_{r+1} \cdots du_d &= \sum_{k_r=0}^{m_r-1} P(U_r = k_r)\, m_r B(m_r - 1, k_r, u_r) \\
= \sum_{k_r=0}^{m_r-1} \frac{1}{m_r} m_r B(m_r - 1, k_r, u_r) = \sum_{k_r=0}^{m_r-1} B(m_r - 1, k_r, u_r) &= \sum_{k=0}^{m_r-1} \binom{m_r - 1}{k} u_r^k (1 - u_r)^{m_r - k - 1} = 1
\end{aligned}$$

for all $u_r \in [0,1]$ which proves that the r-th marginal density of $c_B^{\mathbf{U}}$ is that of a continuous uniform distribution over $[0,1]$, for every $1 \le r \le d$. ◆

Remark 2.1. Note that the line of proof above shows that if $\mathbf{U}=(U_1,\cdots,U_d)$ is a random vector with joint Bernstein copula density $c_B^{\mathbf{U}}$ as above, then also any partial random vector $\mathbf{V}=(U_{i_1},\cdots,U_{i_n})$ with $n<d$ and $1\le i_1<\cdots<i_n\le d$ possesses a Bernstein copula density $c_B^{\mathbf{V}}$ given by

$$c_B^{\mathbf{V}}(u_{i_1},\cdots,u_{i_n})=\sum_{k_{i_1}=0}^{m_{i_1}-1}\cdots\sum_{k_{i_n}=0}^{m_{i_n}-1}P\left(\bigcap_{\ell=1}^{n}\{U_{i_\ell}=k_{i_\ell}\}\right)\prod_{\ell=1}^{n}m_{i_\ell}B(m_{i_\ell}-1,k_{i_\ell},u_{i_\ell}),\ (u_{i_1},\cdots,u_{i_n})\in[0,1]^n.\ \blacklozenge$$

Theorem 2.2. Under the conditions of Theorem 2.1, the Bernstein copula $C_B^{\mathbf{U}}$ induced by $\mathbf{U}$ is explicitly given by

$$C_B^{\mathbf{U}}(x_1,\cdots,x_d):=\int_0^{x_d}\cdots\int_0^{x_1}c_B^{\mathbf{U}}(u_1,\cdots,u_d)\,du_1\cdots du_d=\sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d}P\left(\bigcap_{i=1}^{d}\{U_i<k_i\}\right)\prod_{i=1}^{d}B(m_i,k_i,x_i)$$

for $(x_1,\cdots,x_d)\in[0,1]^d$.

Proof. Let $F_{\mathbf{U}}$ denote the cdf of $\mathbf{U}$, i.e. $F_{\mathbf{U}}(x_1,\cdots,x_d)=P\left(\bigcap_{i=1}^{d}\{U_i\le x_i\}\right)$ for $(x_1,\cdots,x_d)\in\mathbb{R}^d$, and let $\mathbf{Z}=(Z_1,\cdots,Z_d)$ be given by $Z_i:=\dfrac{U_i+1}{m_i}$ for $i=1,\cdots,d$. Then for the cdf of $\mathbf{Z}$, we obtain

$$F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)=P\left(\bigcap_{i=1}^{d}\{U_i\le k_i-1\}\right)=P\left(\bigcap_{i=1}^{d}\{U_i<k_i\}\right)=F_{\mathbf{U}}(k_1-1,\cdots,k_d-1)$$

for $(k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d}T_i$. By applying Lemma 2.1 consecutively d times, it follows that

$$\begin{aligned}c_B^{\mathbf{U}}(u_1,\cdots,u_d)&=\sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1}P\left(\bigcap_{i=1}^{d}\{U_i=k_i\}\right)\prod_{i=1}^{d}m_iB(m_i-1,k_i,u_i)\\&=\sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1}P\left(\bigcap_{i=1}^{d}\{U_i\in(k_i-1,k_i]\}\right)\prod_{i=1}^{d}m_iB(m_i-1,k_i,u_i)\\&=\sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1}\Delta_{m_1,\cdots,m_d}F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)\prod_{i=1}^{d}m_iB(m_i-1,k_i,u_i)\\&=\frac{\partial^d}{\partial x_1\cdots\partial x_d}\mathcal{B}_{m_1}\circ\cdots\circ\mathcal{B}_{m_d}F_{\mathbf{Z}}(u_1,\cdots,u_d)\end{aligned}$$

for $(u_1,\cdots,u_d)\in[0,1]^d$ where $\Delta_{m_1,\cdots,m_d}:=\Delta_{m_1}\circ\cdots\circ\Delta_{m_d}$ is the tensor product of the forward difference operators $\Delta_{m_1},\cdots,\Delta_{m_d}$ from Lemma 2.1 and $\mathcal{B}_{m_1}\circ\cdots\circ\mathcal{B}_{m_d}$ is the tensor product of the Bernstein operators $\mathcal{B}_{m_1},\cdots,\mathcal{B}_{m_d}$ in the sense of [1], section 8.4 (i.e., roughly speaking, the operator with index m_i is applied with the i-th of the d components as a variable and all other components remaining fixed). By integration, we thus obtain

$$C_B^{\mathbf{U}}(x_1,\cdots,x_d) = \int_0^{x_d}\cdots\int_0^{x_1} c(u_1,\cdots,u_d)\,du_1\cdots du_d = \mathcal{B}_{m_1}\circ\cdots\circ\mathcal{B}_{m_d}F_{\mathbf{Z}}(x_1,\cdots,x_d)$$

$$= \sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)\prod_{i=1}^{d} B(m_i,k_i,x_i) = \sum_{k_1=0}^{m_1}\cdots\sum_{k_d=0}^{m_d} P\left(\bigcap_{i=1}^{d}\{U_i < k_i\}\right)\prod_{i=1}^{d} B(m_i,k_i,x_i)$$

for $(x_1,\cdots,x_d)\in[0,1]^d$, as stated. ◆

Remark 2.2. Note that the term $\Delta_{m_1,\cdots,m_d}F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\cdots,\frac{k_d}{m_d}\right)$ in the proof above corresponds – up to an index shift – to the *d*-th order difference of the *d*-increasing cdf $F_{\mathbf{Z}}$,, see, e.g., [17], chapter 6, or [8], Proposition 4.2. For instance, for $d = 2$, we obtain

$$\Delta_{m_1,m_2}F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\frac{k_2}{m_2}\right) = F_{\mathbf{Z}}\left(\frac{k_1+1}{m_1},\frac{k_2+1}{m_2}\right) - F_{\mathbf{Z}}\left(\frac{k_1+1}{m_1},\frac{k_2}{m_2}\right) - F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\frac{k_2+1}{m_2}\right) + F_{\mathbf{Z}}\left(\frac{k_1}{m_1},\frac{k_2}{m_2}\right).\ ◆$$

Remark 2.3. From a probabilistic point of view, in the light of Lemma 2.1, Bernstein copula densities $c_B^{\mathbf{U}}(u_1,\cdots,u_d)$ can also be considered as mixtures of densities of random vectors $\mathbf{Y}(k_1,m_1,\cdots,k_d,m_d) = \left(Y_{(k_1,m_1)},\cdots,Y_{(k_d,m_d)}\right)$ with independent components which follow beta distributions with parameters k_j+1 and $m_j - k_j$ and density

$$f_{Y_{(k_j,m_j)}}(z) = m_j\binom{m_j-1}{k_j} z^{k_j}(1-z)^{m_j-1-k_j} = \frac{1}{\mathrm{B}(k_j+1,m_j-k_j)} z^{k_j}(1-z)^{m_j-1-k_j}$$

for $j=1,\cdots,d$ and $z\in[0,1]$. Here **U** is the mixing random vector. From an algorithmic point of view, this representation is particularly useful for Monte Carlo simulations with Bernstein copulas.

3 Bernstein and checkerboard copulas

There is also a natural relationship between Bernstein and checkerboard copulas as discussed in [2], [5] and [6]. We refer to a slightly more general setup here.

Theorem 3.1 and Definition. Under the assumptions of Theorem 2.1 define the intervals $I_{k_1,\cdots,k_d} := \bigtimes_{j=1}^{d}\left(\frac{k_j}{m_j},\frac{k_j+1}{m_j}\right]$ for all possible choices $(k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d}T_i$. Then the function

$$c_{CB}^{\mathbf{U}} := \prod_{i=1}^{d} m_i \sum_{k_1=0}^{m_1-1}\cdots\sum_{k_d=0}^{m_d-1} p(k_1,\cdots,k_d)\,\mathbb{1}_{I_{k_1,\cdots,k_d}}$$

is the density of a *d*-dimensional copula $C_{CB}^{\mathbf{U}}$, called *checkerboard copula* (induced by **U**). Similarly as before, **U** is called the discrete skeleton of the checkerboard copula**.** Here $\mathbb{1}_A$ denotes the indicator random variable of the set *A*, as usual.

Proof. The assertion is a direct consequence of the fact that a random vector $\mathbf{W}=(W_1,\cdots,W_d)$ follows a checkerboard copula iff the conditional distribution of **W** given **U** fulfills the conditions

$$P^{\mathbf{W}}\left(\bullet\,|\mathbf{U}=(k_1,\cdots,k_d)\right)=\mathcal{U}\left(I_{k_1,\cdots,k_d}\right) \text{ for all } (k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d}T_i,$$

where $\mathcal{U}\left(I_{k_1,\cdots,k_d}\right)$ denotes the continuous uniform distribution over $I_{k_1,\cdots,k_d}$ and

$$\mathbf{U}=(k_1,\cdots,k_d) \Leftrightarrow \mathbf{W}\in I_{k_1,\cdots,k_d} \text{ for all } (k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d}T_i$$

(i.e., **U** denotes in some sense the "coordinates" of **W** w.r.t. the grid induced by $I_{k_1,\cdots,k_d}$). ◆

Remark 3.1. The Bernstein copula induced by **U** can be regarded as a naturally smoothed version of the checkerboard copula induced by **U**, replacing the discontinuous indicator functions

$$\mathbb{1}_{I_{k_1,\cdots,k_d}}(u_1,\cdots,u_d)=\prod_{i=1}^{d}\mathbb{1}_{\left[\frac{k_i}{m_i},\frac{k_i+1}{m_i}\right]}(u_i)$$

by the continuous polynomials

$$\prod_{i=1}^{d}B(m_i-1,k_i,u_i),\ (u_1,\cdots,u_d)\in[0,1]^d.$$ ◆

Theorem 3.2 (Approximation Theorem)**.** Every copula C in d dimensions can be uniformly approximated by a sequence $\left\{C_{CB,r}^{\mathbf{U}_r}\right\}_{r\in\mathbb{N}}$ of checkerboard copulas with grid constants $m_{r1},\ldots m_{rd}\in\mathbb{N}$, if $\min_{1\le k\le d}\{m_{rk}\}$ tends to infinity when r tends to infinity. If C is the cdf of the random vector $\mathbf{Z}=(Z_1,\cdots,Z_d)$ an admissible choice of the discrete skeletons $\mathbf{U}_r$, $r\in\mathbb{N}$ is given by the random vectors $\mathbf{U}_r=(U_{r1},\cdots U_{rd})$ with $U_{rj}:=\lceil m_{rj}\cdot Z_j-1\rceil$ for $j=1,\cdots,d$ where $\lceil z\rceil:=\min\{k\in\mathbb{Z}|z\le k\}$ for $z\in\mathbb{R}$ (rounding upwards). In this case,

$$p_r(k_1,\cdots,k_d)=P\left(\bigcap_{i=1}^{d}\{U_{ri}=k_i\}\right)=P\left(\bigcap_{j=1}^{d}\left\{\frac{k_j}{m_{rj}}<Z_j\le\frac{k_j+1}{m_{rj}}\right\}\right)=P\left(\mathbf{Z}\in I_{k_1,\cdots,k_d}\right)$$

for all $(k_1,\cdots,k_d)\in\bigtimes_{i=1}^{d}T_{ri}$.

Proof. The statement Theorem 3.2 as well as of the following Corollary 3.1 follows from a straight-forward extension of the two-dimensional case discussed in [8], section 5. ◆

Corollary 3.1. Every copula C in d dimensions can be uniformly approximated by a sequence $\left\{C_{B,r}^{\mathbf{U}_r}\right\}_{r\in\mathbb{N}}$ of Bernstein copulas with discrete skeletons and grid constants $m_{r1},\ldots m_{rd}\in\mathbb{N}$, if $\min_{1\le k\le d}\{m_{rk}\}$ tends to infinity when r tends to infinity. The discrete skeletons may be chosen identically as in the checkerboard copula approximation. ◆

The practical importance of Theorem 3.2 lies in the fact that the Monte Carlo simulation of – especially high dimensional – copulas is generally difficult, while a simulation of checkerboard copulas is comparatively easy.

4 Bernstein and rook copulas

In most practical applications, e.g., when modeling financial portfolios containing different stocks and derivatives or insurance portfolios with different types of risk, the stochastic dependence structure of the various model variables is not explicitly known, see, e.g., [9], [13] and [14] for numerous examples. In such situations, assumptions on the class of corresponding (parametric) copula families are sometimes made on the basis of statistical tests. Alternatively, a non-parametric approach could be chosen, for instance identifying the discrete skeleton of a checkerboard or Bernstein copula directly via the observed data. A major problem here is to find a suitable contingency table since the marginal distributions must be discretely uniform, which means that a set of side conditions has to be fulfilled. Also, this approach becomes ineffective for higher dimensions *d*, since in general $\prod_{i=1}^{d} m_i$ real numbers have to be stored in order to describe the distribution of the discrete skeleton completely. Such problems are completely avoided if so-called rook copulas are used for modelling the discrete skeleton.

A rook copula is a particular checkerboard copula with the same grid size in each dimension that distributes probability mass according to the placement of rooks on a checkerboard without mutual threatening. It can in general be constructed in *d* dimensions as follows. Let

$$M := \begin{bmatrix} \sigma_{01} & \sigma_{02} & \cdots & \sigma_{0,d-1} & \sigma_{0d} \\ \sigma_{11} & \sigma_{12} & \cdots & \sigma_{1,d-1} & \sigma_{1d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \sigma_{m-2,1} & \sigma_{m-2,2} & \cdots & \sigma_{m-2,d-1} & \sigma_{m-2,d} \\ \sigma_{m-1,1} & \sigma_{m-1,2} & \cdots & \sigma_{m-1,d-1} & \sigma_{m-1,d} \end{bmatrix}$$

denote a matrix of permutations in column vector notation, i.e. each $(\sigma_{0k}, \sigma_{1k}, \cdots, \sigma_{m-1,k})$ is a permutation of the set $T := \{0,1,\cdots,m-1\}$ for $k = 1,\cdots,d$. A checkerboard copula *C* is a rook copula iff there holds

$$p_m(k_1,\cdots,k_d) = P\left(\bigcap_{i=1}^{d}\{U_i = k_i\}\right) = \frac{1}{m} \Leftrightarrow (k_1,\cdots,k_d) = (\sigma_{t1}, \sigma_{t2}, \cdots, \sigma_{t,d}) \text{ for some } t \in T.$$

The distribution of the discrete skeleton of a rook copula can thus be completely described by storing just $m \cdot d$ instead of m^d real numbers.

Example 4.1. The rook copula corresponding to the picture on the right is given by the matrix

$$M = \begin{bmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 0 & 1 & 4 & 2 & 3 & 6 & 5 & 7 \end{bmatrix}^T. \quad ◆$$

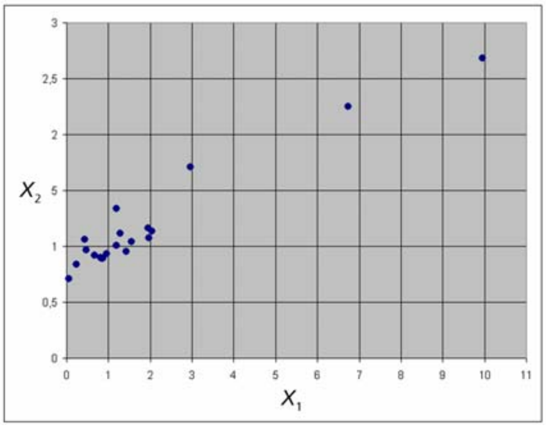
In practical applications, in the case of continuous distributions, the permutation matrix pertaining to a rook copula can directly be extracted from the ranks of the observed random vectors according to the following procedure. Given a matrix $\mathbf{x} = \left[x_{ij}\right]$ of data, where $i = 1,\cdots,n$ is the i-th out of n independent d-dimensional observation row vectors and $j = 1,\cdots,d$ is the corresponding component (dimension) index:

- For each j, calculate the rank r_{ij} of the observation x_{ij} among $x_{1j},\cdots,x_{nj}$ for $i = 1,\cdots,n$.
- Form the matrix $M := \left[\left(r_{ij} - 1\right)\right]$ of permutations for the empirical rook copula.

W.r.t. Monte Carlo simulations, it is extremely easy to generate samples that follow either a rook copula or a Bernstein copula with the same discrete skeleton. For simplicity, we explain the procedure by means of the following example only.

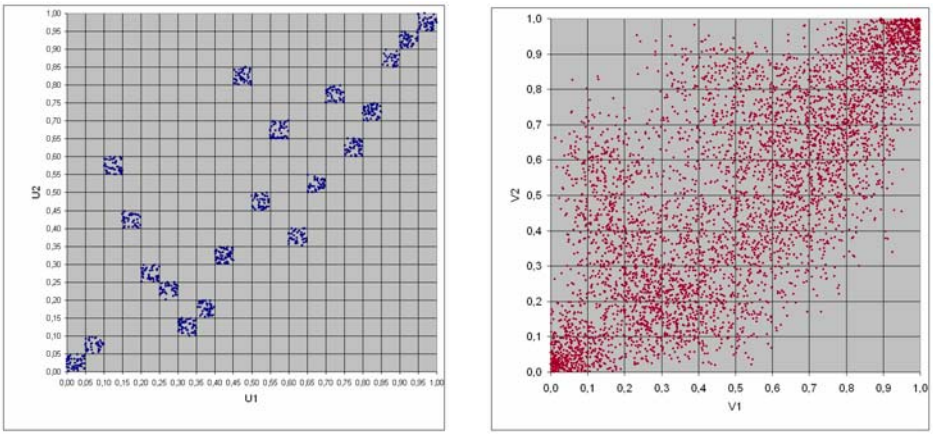
Example 4.2. The following table contains some original data $\left(x_{i1}, x_{i2}\right)$, $i = 1,\cdots,20$ from an insurance portfolio of storm and flooding losses, observed over a period of 20 years, their ranks and the permutation matrix *M*.

i	x_{i1}	x_{i2}	r_{i1}	r_{i2}	M	
1	0.468	0.966	4	9	3	8
2	9.951	2.679	20	20	19	19
3	0.866	0.897	8	4	7	3
4	6.731	2.249	19	19	18	18
5	1.421	0.956	13	8	12	9
6	2.040	1.141	17	15	16	14
7	2.967	1.707	18	18	17	17
8	1.200	1.008	11	10	10	9
9	0.426	1.065	3	12	2	11
10	1.946	1.162	15	16	14	15
11	0.676	0.918	5	6	4	5
12	1.184	1.336	10	17	9	16
13	0.960	0.933	9	7	8	6
14	1.972	1.077	16	13	15	12
15	1.549	1.041	14	11	13	10
16	0.819	0.899	6	5	5	4
17	0.063	0.710	1	1	0	0
18	1.280	1.118	12	14	11	13
19	0.824	0.894	7	3	6	2
20	0.227	0.837	2	2	1	1



Scatterplot of observed risks x_{i1} and x_{i2}
(in million euros)

In the first step, we draw a pair $\left(\sigma_{i1}, \sigma_{i2}\right)$ out of *M* with equal probability $\frac{1}{m} = \frac{1}{20}$ w.r.t. the index $i \in \{0,\cdots,m-1\} = \{0,\cdots,19\}$. In the second step, we either draw a sample $\mathbf{Z} = \left(Z_1, Z_2\right)$ from a continuous uniform distribution over the rectangle $I_{\sigma_{i1},\sigma_{i2}} = \left[\frac{\sigma_{i1}}{m}, \frac{\sigma_{i1}+1}{m}\right] \times \left[\frac{\sigma_{i2}}{m}, \frac{\sigma_{i2}+1}{m}\right]$ for the rook copula, or a sample $\mathbf{Z} = \left(Z_1, Z_2\right)$ with independent components where Z_j follows a beta distribution with parameters $\sigma_{ij} + 1$ and $m - \sigma_{ij}$, $j \in \{1,2\}$.



5000 simulated random vectors
following the rook copula (left) and the Bernstein copula (right)

A generalization of the procedure to arbitrary dimensions, replacing the rectangle $I_{\sigma_{i1},\sigma_{i2}}$ by a general cube, is obvious. ◆

Note that according to a fundamental theorem in statistics, the empirical distribution function of a multivariate observation converges uniformly to the true cdf when the sample size increases. Likewise, the empirical copula based on the extracted marginal ranks converges uniformly to the true underlying copula. This implies that with an increasing number of observed data, the rook copulas as well as the Bernstein copulas with the discrete skeletons derived from the marginal ranks converge to the true underlying copula as well, since in both cases the grid constant m corresponds to the sample size.

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