# ACC513
# Management Accounting for Decisions

FACULTY OF BUSINESS

**Study Guide**

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*Author*
**Professor V Fatseas**

*Educational designer*
**Deborah Murdoch**

Produced by Division of Learning and Teaching Services, Charles Sturt University, Albury - Bathurst - Wagga Wagga, New South Wales, Australia.

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# Preface

## How to use the Study Guide

# Using the Study Guide effectively

The objectives of this *Study Guide* are to facilitate your progress through each of the topics and sub-topics in the subject and to ensure that you have acquired the requisite understanding and knowledge of the concepts and techniques presented in each of the topics. Your expertise in management accounting and your ability to apply the knowledge gained from study of this subject are tested in the study tasks and review tasks, the compulsory assignments and final examination.

You should use the Study Guide on an interactive basis and be guided in approaching and sequencing your studies for a topic and its sub-topics by the activities presented. For each topic, the *Study Guide* contains:

1. **Essential reading –** the required readings for the topic from the prescribed textbook and from the Reading section of your Study Guide.

2. **Objectives** - the purpose of these is to clearly identify the concepts and procedures you should have mastered having read the required readings and attempted the study tasks. The learning objectives are task-oriented, clearly identifying which aspects of the topics you need to be able to describe, discuss, calculate and/or apply.

3. **Commentary –** the Commentary section provides a discussion of the content covered in each topic and sub-topic, highlights some of the key issues in topics and provides further information or examples on the more problematic aspects of a topic. The Commentary also functions to show you how various parts of a topic and different topics fit together. The Commentary is supplementary to the required readings and is *not* a substitute for them. The Commentary section includes a range of activities that you need to perform including Reading, Study Tasks, From Theory to Practice activities and Review Tasks.

4. **Summary –** this section presents a brief recap of the topic and directs you to a Review task that follows.

5. **Self-test exercises –** these questions are grouped together at the end of every topic presented in the Study Guide. The questions can include a selection of multiple choice, completion statements, True/False and other questions from the **Question Bank.** The questions appear as Study Task and Review Task activities within the Commentary.

6. **Self-test solutions –** fully worked solutions to each of the Self-test questions, including those found in the Question Bank. The solutions are designed to provide you with immediate feedback on your self-testing.

7. **References –** full citations for references made in the Commentary to books and journal articles, other than the set textbook or references provided in that textbook.

## Facilitating your learning

As a guide, when you begin a new topic, consult the Objectives section first, then read the Commentary section until you come to an activity. Usually the first activity will be a requirement for you to Read certain pages of a chapter in the text. Complete this reading, then go back to the Commentary until you reach another activity. This next activity might require you to complete a Study Task, which will mean preparing an answer to one or more of the Self-Test Questions. Once you have satisfactorily completed the activity, resume reading the Commentary until you come across the next activity and so on.

However, depending on your learning style and personal preferences, you might not wish to attempt the study tasks as indicated by the Commentary. That's not a problem, provided you attempt the self-test exercises at the end of each topic. In doing so, it should help you monitor your progress in the subject.

You should also make notes based on the Readings, Commentary and Self-test questions. The learning objectives and commentary for each topic should assist and direct you in your note-taking.

# Part 1

## Models of choice, probability calculations and distributions

# Topic 1 Strategic management accounting and choice processes

## Objectives

At the end of this topic, you should be able to:

- explain why management accounting has taken on a strategic emphasis;
- describe the general principles of rational decision making;
- distinguish between normative and descriptive models;
- identify alternative models of choice and provide examples of situations when each model is an appropriate description of actual decision processes.

## Commentary

In Topic 1 we begin by briefly reviewing the history of the development of management accounting. We show how changes in the business environment have led to the development of new management accounting techniques and practices to support strategic decision making. The rational decision model is introduced as the ideal approach to decision making, together with some alternative models of a more descriptive nature of actual choice processes in organisations. The case is made for managers to aim to be as rational as possible in making decisions.

### Development of management accounting

The development of management accounting in the USA has been comprehensively described by Johnson and Kaplan (1987). They explain how the onset of the Industrial Revolution in the nineteenth century provided opportunities for entrepreneurs to benefit from economies of scale by combining and managing separate processes of production. For example, the textile industry consisted of the processes of spinning, weaving and finishing. Prior to the Industrial Revolution each of these had been carried out as separate businesses whose finished output was sold to the next operator in the chain. Now entrepreneurs could increase their profits by combining these processes within the one business. In order to manage such enterprises which combined two or more processes under their control, management had to hire workers. Wage contracts replaced piece rates, and production costs were accumulated, not so much to calculate the unit costs of production, but to control conversion costs and to motivate and evaluate the performance of supervisory managers.

The US railroads and steel mills furthered the use of management accounting information for cost control purposes. Managers of steel mills collected the costs of each order (i.e., job costing), while the railroads calculated the costs per ton-mile. In addition, because of the size of the railroad operations and task specialisation, division of management was introduced. This led to the use of cost information to assess the performance of these subordinate managers. The growth of large retail store chains also promoted the use of measures of performance of departmental managers, measures such as departmental gross margin and the rate of stock turnover.

With the advent of the manufacture of complex, machine-made goods (eg, sewing machines, typewriters, reapers) came a need for closer control over specialised workers. Material and labour usage and costs were compared with standards, and variances analysed. In the early twentieth century there began to appear diversified companies such as Du Pont and General Motors, engaged in more than one product line or activity. Budgets were designed and used to co-ordinate activities and to promote goal congruence within these organisations. The Du Pont Company developed the concept of ROI (return on investment) as a measure to evaluate the performance of managers who were held responsible for the profitable use of capital.

Most of the management accounting techniques commonly in use until the 1980s were developed during the period 1825-1925. Moores (1992) claims that 'This was arguably management accounting's golden age'. From 1925 to about 1985 was a period of relative stagnation in the development of management accounting. The need for financial accounting grew because of the use of capital markets as a vehicle for providing equity capital for corporations and the resulting requirements to produce audited financial statements. These financial reporting requirements influenced the allocation of historical transaction data to cost of goods sold and inventories. They also resulted in the use of crude overhead allocation procedures usually based on direct labour usage or cost. The one significant development in this period was capital budgeting. Mechanisation, arising as a response to the production of standardised products, resulted in the need for increased capital investment. In order to evaluate proposed capital investments capital budgeting techniques were developed in the 1950s.

By the mid 1980s there was emerging a significant change in the business environment. To survive this change firms had to become more competitive, requiring management accounting to provide new types of information for competitive support.

## Changing business environment

Three periods of economic evolution in the Western world have been identified by social scientists: the *agricultural era*, the *industrial era*, and the *information age*.

The agricultural era lasted for thousands of years, from around 8000 BC to the mid-eighteenth century. This era was based on agriculture in which any wealth was achieved by physical labour.

The agricultural era eventually gave way to the industrial era in which machinery replaced muscle power, and factories replaced agriculture as the dominant employer. Blue-collar workers operated equipment in crowded factories. Developed economies were dominated by manufacturing. During the 1950s and 1960s there was prolonged growth. Domestic markets were protected from competition and thus there was lack of incentive on the part of many firms to maximise efficiency. High costs could be passed on to customers who had little alternative but to pay prices which were higher than they might otherwise be under more efficient productive operations.

The period since the 1960s has been called the *post-industrial society*, or the *global village*, or maybe the most accurate term is the *information age*. Blue-collar workers were increasingly replaced by white-collar workers who work individually or in small teams using computers in office environments. The information age has been characterised by a relative decline in employment in manufacturing compensated for by a massive expansion in service sector employment. Although employment in manufacturing has declined, this has not meant that post-industrial societies are manufacturing less. They are actually producing, more. The progressive use of modern technology and production processes (automation, robotics, computer integrated manufacturing) has reduced the number of people needed in manufacturing industries which have become capital intensive. The largest employment in the Australian economy is now in the service sector, contributing approximately 70% of GDP, with agriculture contributing only 4%, and industry 26%.

Since the 1970s national economic boundaries have virtually disappeared as the world has moved towards a global economy. The rise of multinational corporations and the reduction in protective tariff barriers has resulted in increased international trade and competitiveness. Australia was a relatively late entrant to the global economy. The 1983 deregulation of the financial services sector and the floating of the Australian dollar, together with the progressive reduction of import tariffs, completed Australia's projection into the international economy.

As a result of the reduction in tariff protection in many industries, Australian businesses were forced to become competitive or else lose their domestic market share to foreign competitors. We have seen the Australian motor vehicle industry subjected to overseas imports priced competitively to compete with locally produced cars. The Holden Commodore and the Ford Falcon are slipping in their dominance of the market. Automotive component suppliers have also suffered as Australian car manufacturers source their supplies globally. Deregulation of telecommunications saw overseas companies such as Optus (now taken over by Singapore Telecommunications) enter to compete with Telstra. Virgin Blue challenged Qantas and Ansett. Some firms did not survive – Ansett for example failed.

Some Australian companies were taken over by overseas companies. For example, San Miguel Corp from the Philippines took over National Foods in 2005 (producer of cheeses, Yogo, Pura Milk, Big M and Vitasoy) and then integrated it with other Australian companies that San Miguel Corp had taken over (Tasmanian brewery James Boag & Sons and the Berri fruit juice company).

Other Australian businesses grew through takeovers and mergers and expanded operations into the global marketplace. The challenge for Australian businesses is to gain and maintain a competitive advantage. This involves making strategic decisions as part of what we call the **strategic management** process.

### From theory to practice

What are your thoughts on how management accounting has evolved? Do you think management accountants would be in a better position if the emphasis of management accounting was more strategic? Why/why not?

## Strategic management accounting

The challenge for organisations today is to gain and maintain a competitive advantage. To this end, organisations must engage in strategic management. Strategic management is concerned with making and implementing strategic choices in order to attain and sustain superior financial performance.

In order to assist managers in their endeavours, management accounting has had to take a more strategic focus. It is now concerned with providing financial and non-financial information to assist managers in their strategic choices.

The objective of strategic management is to 'attain and sustain superior financial performance' (Lewis 1993). Accordingly, strategic management involves strategic choices concerning:

- what business to be in
- how to compete in that business
- how to organise physical, financial and human resources to facilitate the achievement of corporate goals

**Strategic management accounting is concerned with supporting strategic management activities.** It provides financial and non-financial information to assist managers in:

- developing new competitive strategies
- evaluating existing competitive strategies
- monitoring and assessing progress towards particular strategies

The focus of strategic management accounting is broader than traditional management accounting. Traditional management accounting tends to be **internally focused**. As Shank and Govindarajan (1992) explain, traditional management accounting takes a **value-added** approach, starting from payments to suppliers for purchases and stopping with receipts from customers for sales. The conventional view is that the way to influence profits is by maximising the value added between purchases and sales.

According to Shank and Govindarajan (1992) there are two major flaws in this type of thinking. First, starting with purchases is too late because it misses opportunities for advantageous links with suppliers, links which may benefit both parties. For example, quality manufacturing practices by suppliers ensure better quality materials to manufacturers and at the same time may help suppliers develop additional markets or even export markets. Consultation between supplier and manufacturer can help in mutual solving of particular problems in manufacture. New delivery techniques can help reduce costs for both parties, eg deliver bulk chocolate in liquid form in tankers rather than in moulded bars which have to be unpacked and melted (Porter, 1985).

Second, value added analysis stops too early because it misses opportunities for links with customers. For example, producers can locate close to customers so as to reduce delivery costs, eg, a container manufacturer decided to locate next to a brewery and delivered containers via overhead conveyers directly onto the customer's assembly line, reducing costs for both supplier and customer (Hergert and Morris, 1989).

Strategic management accounting adopts this broader focus external to the firm in addition to the conventional internal focus. Each firm is examined from the viewpoint of the whole chain of value-creating activities from basic raw materials to end-use consumers. The firm may only be a part of this large set of activities in the chain. But to gain and sustain a competitive advantage the firm must understand the whole system, not just the part of the chain in which it participates. Strategic management accounting is more customer-focused because it is the patronage of customers which provides sales and profits. Strategic management accounting is concerned with strategic decisions which deliver competitive advantage to enable the firm to survive and to produce superior financial performance.

**Study task**

Check your understanding of management accounting history and the development of strategic management accounting by attempting the following Self-test questions:

- Matching questions 1-3
- Completion statement questions 1-2
- Multiple choice questions 1-3

Check your answers in the 'Self-test solutions' section.

The making of strategic decisions requires the acquisition of decision making skills and techniques, as well as access to appropriate information. The management accountant is expected to analyse strategic opportunities and hence needs a knowledge of decision making techniques and processes.

## Choice processes and models

The terms *decision making process* and *choice process* are synonymous. The choice process usually consists of a number of ordered steps until a choice or decision is made. Choices can be made by individuals, groups or organisations. Once we move from individual choice to group choice the process becomes more complicated because of the need to have consensus. A model of choice refers to the way in which decisions are made. Models of choice can be normative (this is the way you should go about decision making) or descriptive (this is the way we see people making decisions)

The process of choice is often referred to as decision making, the act of choosing a preferred action (or strategy) from an array of alternatives. Decision making is a pervasive activity to which most people devote a good deal of time. Frequently choices are made intuitively, with little conscious effort, or sometimes instinctively. Choice outcomes may be relatively unimportant, for example whether to take a shower or a bath, or they may be instinctive responses involving skills acquired through repetitive practice, as in driving a car. On the other hand, major personal or business decisions often involve an agonising process of weighing alternatives and their possible outcomes; for instance, which university course to undertake, or which house to buy, or which business strategy to adopt.

Choice is the end product of the decision making process. It is possible to view the decision making process as a series of related steps or stages. Simon (1977) suggests that decision making comprises three principal phases[1] which he calls intelligence, design and choice.

### 1. Intelligence phase

In the intelligence phase, the opportunity (or necessity) for decision making is sought by scanning the environment. People actively search for opportunities to exercise choice by monitoring their environment for problems or surprises. Awareness of the existence of a problem or opportunity is a necessary precondition for decision making activities to be put into motion.

### 2. Design phase

In the design phase, possible courses of action are determined and analysed as potential solutions to the problem detected or as strategies to take advantage of perceived opportunities. Unless there exist more than one possible action there is no choice to be made. Alternative strategies have to be discovered or invented - they do not just appear. As each alternative is generated its consequences are explored. For example, if we adopt action x what will be the likely impact on our profits next year?

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1 Simon (1977) also identifies a fourth phase, *review*, which is a process of evaluating past choices.

**3. Choice**

The third phase, choice, involves the selection of a particular course of action, from the available alternatives identified in the design phase, for implementation. Such a selection process is based on the evaluation of each alternative and some preference ranking of the effectiveness of each alternative relative to the achievement of objectives. Thus there is a requirement for some kind of choice criterion by which preferences can be explicated and ordered.

Choice processes, that is the way in which decisions are made, depend partly on the nature of the choice arena. Decisions may be made by individuals, or by groups of individuals. The group may be simply an informal collection of people (e.g. a local tennis club) or members of a formal organisation (e.g. the divisional managers of a large diversified company). We could refer to these choice contexts as individual choice, group choice and organisational choice respectively. An individual may adopt different choice processes for problems of differing complexity and of differing importance. Different individuals facing an identical problem may follow varying processes in arriving at their choice. They may have different goals or objectives, they may perceive different alternatives, they may have different levels of available resources and they may use different methods for selecting their chosen alternatives.

Group choice adds complexity to the choice process because different group members will possess different individual goals and because of disparate levels of power and persuasiveness among the group constituents. Because of varying individual goals many group decisions may necessitate a reconciliation of the goals of the various participants, often leading to compromise goals and perhaps unresolved conflict among individuals. Differential individual power and charisma can be used to influence participants and may be used to foster support for the views of dominant individuals. Organisational choices can also be taken amidst goal conflict and the political machinations of individual participants. In addition, organisations feature hierarchical, delegated authority, rules, policies and procedures all of which can impact the decision process.

## Normative and descriptive models

A model is a simplified representation of some part of reality. It is an abstraction of a real-life phenomenon. A model is usually simplified because reality is too complex to copy exactly and much of the complexity is irrelevant for the modeller's purpose. Models have been found to be useful aids in many areas of study. Wind tunnels are used on models of new aircraft to assess flying performance. Monkeys and dogs were used as analogue models of man in early space flight. A cash budget is a model of a firm's inflows and outflows of cash. Perfect competition is a micro-economic model used to determine optimum price and output levels under certain restrictive assumptions. Choice models are representations of how choices should be, or are actually made.

The purpose of building choice models has been twofold. First, **normative** or **prescriptive** models have been constructed in order to show a decision maker how (s)he *should* make a decision, or a class of decisions. A large part of the research in decision making has been essentially normative. The statistical decision theorist and the economist have tried to construct models which tell how decisions should be made.

Because observed decision-making behaviour frequently departs from normative guidelines **descriptive** models have been constructed in an attempt to simulate or describe actual behaviour. The psychologist or behavioural scientist has tried to explain how decisions are made in practice. The following example (borrowed from Taylor, 1970) illustrates the difference between a normative and descriptive model of a choice process.

**From theory to practice**

Suppose an urn contains 100 marbles, 75 red and 25 black. A marble is drawn at random from the urn and the audience is asked to record what colour it believes the marble to be. The marble is then shown, replaced, and the urn's contents again randomised. The aim of the subjects is to maximise the number of correct guesses.

What strategy would you employ?

The normative approach to the task is to build a model which tells a decision maker how to maximise the number of correct responses:

Let C = the number of correct guesses,
N = the total number of guesses,
B = the number of times black is recorded.
Then (N-B) = the number of times red is recorded.

The probability of being correct if you call black = 25/100, or 1/4
The probability of being correct if you call red = 75/100, or 3/4

On average, $C = 3/4(N-B) + 1/4B = 3/4N - 1/2B$

In order to maximise C one should minimise B, and this means calling red every time.

The descriptive approach is to observe human behaviour and try to explain or predict such behaviour on the part of subjects. When the above experiment has been conducted most people do not exhibit the behaviour which the normative analysis demonstrates to be optimal. What they usually do is to call red and black approximately in the proportions of 3 to 1.

**Study task**

What are the stages in the decision making process. What is the difference between normative and descriptive models? Try the following Self-test questions:

- Matching question 4
- Completion statement questions 3-5

**Rational model of choice**

The rational model of choice describes how decisions *should* be made - it is a normative model. It has fairly restrictive assumptions, but nevertheless rational decision making is a worthwhile aim.

The rational model is regarded as a normative model, one which prescribes the best way to make choices. It is based on the concept of economic man, a rational, completely informed, single decision maker. The decision maker has a single goal of maximising expected utility. Given a problem, (s)he identifies all possible alternative solutions. These alternatives are produced by a search process which in the classical rational model is assumed to be costless. The likely outcomes or consequences of each possible action (or alternative, or strategy) are assessed. These consequences are frequently measured in terms of costs or benefits. If there is uncertainty with respect to the outcomes for any action then the decision maker estimates the probability of occurrence of each of the various consequences. A rational choice involves selecting that action which maximises goal achievement. Hence the decision maker ranks the consequences of each alternative and selects the alternative with the preferred consequence. In practice this means selecting the alternative which maximises expected profit or minimises expected cost, or more generally, maximises expected utility. The rational model, to be applicable to organisations, requires agreement on a common goal or consistent set of goals, implying a centralised system of power and control.

The following scenario gives you an example of the rational model:

Suppose that Bill, a farmer, is faced with the problem of what crop to plant in the present growing season. His soil is suitable for only two crops, wheat or sunflower seeds. He estimates the following net returns from each action:

| Action | Consequence |
|---|---|
| $a_1$: Plant wheat | $100 000 |
| $a_2$: Plant sunflower seeds | 60 000 |

Given the certain outcomes, Bill would plant wheat to maximise his profits. Actually the rational model requires that these profits be transformed into measures of utility and he should select the action with the maximum utility. If we assume, however, that the decision maker is risk neutral (i.e. has a linear utility function) then such a transformation is unnecessary.

Suppose the outcomes are uncertain, that the net return from each crop is dependent on the weather experienced during the growing season. Suppose that there are two possible states, wet weather or average weather. Then Bill has to estimate the likely consequences of each state on each action. For example:

| **State:** | $s_1$: *Wet weather* | $s_2$: *Average weather* |
|---|---|---|
| **Action** | | |
| $a_1$: *Plant wheat* | \$40 000 | \$100 000 |
| $a_2$: *Plant sunflower seeds* | 120 000 | 30 000 |

Thus wheat would return \$100 000 in average weather conditions but only \$40 000 in a wet season. Sunflower seeds tend to produce a better return in wet weather (\$120 000) than in average weather conditions (\$30 000). Because the seasonal weather is uncertain, the rational model also requires Bill to estimate the probability of each state occurring and base his decision on expected values. Suppose that he estimates (on the basis of meteorological records) that there is a probability of 0.25 of wet weather and 0.75 of average weather. Then the expected profit from wheat is \$40 000(0.25) + \$100 000(0.75) = \$85 000 and from sunflower seeds is \$120 000(0.25) + \$30 000(0.75) = \$52 500. Hence the optimal choice is to plant wheat because it has the higher expected profit.

## From theory to practice

Can you identify some constraints in the rational model? Jot down your answer below and compare with the discussions that follow.

### *Criticism of the rational model*

There have been numerous criticisms of the rational model. First, the assumptions are too unrealistic. The decision maker must discover **all possible alternatives** and needs to be able to assess **all the consequences** of all of these alternatives. This search process is assumed to be **costless**. Second, people have **limited cognitive abilities**. Even if all the possible alternatives are identified and all consequences estimated the cognitive limitations of humans prevent them from being able to assimilate and evaluate all the information. Third, there seems to be **little descriptive support** for the rational model. One does not observe 'rational' decision making behaviour in action. Finally, the model **lacks organisational relevance**. In organisational decision making there are too many participants with varying perceptions and goals to be able to gain a consensus on the framing of the decision and selection of a preferred alternative.

Nevertheless, the rational model can be a valuable conceptual tool, just like the perfect competition model in economics. The model provides an idealistic view of decision making and defines the logic of optimal choice. It is held by some as a prescriptive norm, something to aim for in decision making endeavours. Others, who adopt a more pragmatic approach and view the rational model as being quite unrealistic in its requirements, still find it useful as a basis for comparison with alternative models which they promulgate as possessing more descriptive realism.

**Study task**

Test your understanding of the process of the rational model. Would you know how to calculate the consequences of each action? Go to the Self-test questions section and try the following:

- Matching question 5-8
- Multiple choice questions 4-5

**Alternative models of choice**

Various alternative models have been suggested as more accurate descriptions of decision making in practice - they are descriptive models. None of these models is completely descriptive of actual choice processes, but each has features which are found in all organisations. We can learn from what each model suggests. The following models are representative of some of the major alternatives found in the literature.

***Satisficing (or bounded rationality) model of choice***

Simon (1955) began the demolition of the rational model, focusing on the comprehensive knowledge and analysis implied in this approach. He argued that it was not practical to generate all the relevant alternatives; that only very few of all the possible alternatives ever come to mind. Even if all the alternatives could be established the task of anticipating the consequences of those alternatives and attaching values to them is not possible because of fragmentary knowledge and lack of experience. Because of the cognitive limitations of the human mind the ability to formulate and solve complex problems with 'objective rationality' is beyond a human's capacity. Consequently Simon rejected the comprehensive rational model as lacking in descriptive reality, and concluded that rather than optimise people **satisfice** - find a solution which is good enough. He suggested that people are intentionally rational but the best they can do is to exercise **bounded rationality**; people put bounds on their search for alternatives and limit themselves to a manageable few. Having constructed a simplified model after a moderate search for a limited number of alternatives and their consequences man then behaves rationally with respect to this model.

In practice, such an approach might begin by searching for possible alternative courses of action and for information about the consequences of each alternative. The first alternative which meets some minimum standard of satisfaction with respect to goals is selected - that is, the decision maker satisfices. Such bounded rationality or satisficing often involves the use of heuristics - rules of thumb which give solutions that are good enough most of the time. For example, in attempting to set an advertising budget, a simple heuristic such as set expenditure equal to 5% of sales might be employed. Clearly such a strategy is far from optimal, especially considering that advertising should influence sales, but nevertheless such a rule is often adequate and saves immense search and analysis.

Advocates of the 'rational' model of choice usually imply Simon's version of bounded rationality by limiting the number of alternatives for consideration but then selecting the best alternative from the set by 'optimising' in the chosen domain. Because not all possible alternatives and consequences are considered such a procedure does not produce a global optimum but rather a local optimum which might be termed a satisficing solution.

***Organisational procedures model of choice (SOPS)***

While Simon's satisficing model can be viewed as an individual model of choice it can have organisational applications. For example, many managers are satisficers. Specific models, however, have been proposed within an organisational context, often as extensions of the satisficing model. One such model emanates from the work of Cyert and March (1963) who focused on the effects of organisational structure and conventional practice on the formulation of goals and expectations and the execution of choice. The assumption of consistent, overall organisational objectives, necessary for the rational model, is removed. Rather, they viewed an organisation as consisting of a number of coalitions, each of which has its own priorities, goals and focus of attention. Bargaining among coalition members produces agreements that impose constraints which may be viewed as goals. Problems are identified as coming within the province of a particular coalition or sub-unit of the organisation and are solved by the application of standard operating procedures (SOPS) or rules. Each problem involves the application of 'problemistic search' which involves either a search for an appropriate rule in repetitive situations ('What did we do last time?') or else the development of a satisfactory new rule in novel situations. Organisations rely on habitual ways of handling problems. The SOPS form precedents for action and form part of the organisation's memory of procedures learned in previous choice processes. When feedback indicates that a standard rule has ceased to be satisfactory a new procedure is created, a process of organisational learning.

An example of the organisational procedures or SOPS approach to budgeting is the use of some simple rules for developing next year's budget, e.g. increase this year's figures by 10%, associated with some rules for handling requests for larger increases. By contrast, the rational approach, as exemplified in zero-based budgeting, regards this year's allocations as largely irrelevant and requires fresh justification for next year's requests.

Undoubtedly the use of SOPS is found in most organisations and can account for some of the choice processes. Routine problems can often be best handled by the application of standard policies or rules, thus relieving organisational members from 're-inventing the wheel' and providing consistency and uniformity of treatment in similar situations. Such a choice process, however, does not seem relevant to one-off, non-routine problems, or strategic decisions. Other models may better explain non-recurring decision problems and the choice processes employed in their solution.

***Political model of choice***

Many organisational decision making opportunities, especially at the strategic policy level, are subject to the exercise of power by influential individuals. The capacity of an individual to exercise power is dependent on access to sources of power. Handy (1981) identifies the following sources of power: physical power (e.g. a dictatorial boss), resource power (from control of resources desired by others), position power (arising from one's position in the organisation), expert power (as a result of acknowledged expertise) and charisma. Individuals in organisations have opportunities for exercising power to differing degrees in different situations.

In the political view of organisational choice (see, for example, Pfeffer, 1981) conflict is regarded as normal and power and influence often determine the outcome of decisions. Action results from bargaining and compromise, so that decisions seldom reflect the preferences of any individual or group. Individuals with the greatest power receive the greatest rewards. An advocate of a particular policy or alternative must attempt to build a consensus of support for his view. If there are rival policies or advocates there is competition among individuals to gain support for their policies and all the techniques of politics appear - persuasion, bargaining and accommodation. Any particular choice outcome depends upon who participates, what determines each advocate's stand on the issues involved, the source of each participant's power and how power is used.

Clearly the use of power is apparent in organisational choice processes. Its exercise can be observed in the quashing of issues and opposition by the use of such devices as the setting of the choice criterion, strategic presentation or withholding of information and the introduction of pseudo-participation in decision making. Power is sometimes used overtly in organisational decision making, but more commonly its use may be difficult to observe directly because its practitioners clothe their manipulations in legitimate procedures. The use of committees is a respectable and democratic procedure whereby decisions can be ratified. Responsibility for a decision is diffused when a committee is used, so that no single person is seen to be responsible. A powerful member or group can control the composition of the committee, and set the agenda. Operating within a limited time frame, under specified rules, with the choice alternatives already delineated and with information controlled it is unlikely that a committee decision will vary from the result desired. The committee members' participation (or pseudo-participation) in the decision process increases the likelihood of general acceptance of the resulting choice, so that carefully manipulated committees can become rubber stamps for the policies of those with the power to covertly achieve their ends in a seemingly legitimate and democratic manner.

### *Disjointed incrementalism*

Lindblom (1959), in discussing public policy planning, focused on the realities that make rationality impossible. He asserted that clarification of objectives founders on social conflict; that the required information for the rational approach is not available, or only at prohibitive cost; and that problems are often too complex for man's finite intellectual capabilities. Lindblom argues that the decision maker should avoid comprehensive analysis and rely on the strategies of 'successive limited comparisons', or disjointed incrementalism. That is, the analysis of a problem should be limited to alternatives that differ only incrementally from existing policy, thus reducing the number of alternatives and drastically simplifying the analysis. Objectives are not pre-defined; what is feasible may define what the goals should be. Policy making, then, is remedial; it is a process of 'muddling through' by means of small adjustments at the margin away from existing policy when it is seen to be unsatisfactory, rather than a move towards predetermined objectives. A continuous succession of incremental changes can overcome the need for rational analysis.

Disjointed incrementalism can be successful when there is rapid feedback from community watchdogs who hold decision makers accountable, and there exists the ability to easily adjust or reverse decisions with unfavourable consequences. It would be inappropriate when feedback is not rapid, or there are no powerful lobbies to force the decision maker to adjust. The decision to prescribe thalidomide to pregnant women for morning sickness is an example of unfortunate consequences of 'muddling through'. Because feedback was slow (it took years to establish the causal connection between thalidomide and deformed babies) and the consequences were severe, such a decision was quite a mistake.

Lindblom's model applies most directly to public policy planning but is relevant for organisational decision making because many managers are incrementalists.

### *Anarchical model of choice*

The anarchical model (also known as the artifactual or decision process model) represents a large departure from the rational model and is said to be representative of choice processes in an 'organised anarchy' (Cohen, March and Olsen, 1972). In such a setting there are no predefined preferences held by participants, no clear overall organisational goals and no powerful individuals with defined preferences. Goals are formulated after a choice is made in order to explain, justify or rationalise the decision. Individuals, having made a choice, set about justifying it and in doing so discover appropriate goals which, if pre-existing, would have led to the alternative selected under some form of rationality - people like to appear to have made rational decisions.

Choice is the result of the situation in which participants become located. Individuals become participants because they are under some obligation to participate, or have the opportunity to do so and have nothing more pressing to attend to. Individuals often have, however, competing demands on their precious time and have to be selective as to what meetings or decision processes they participate in. Solutions to problems are determined by who happens to participate in a choice situation, the particular problems that are on the agenda and the solutions put forward then and there by those participating. Chance, as much as deliberation, may determine particular decision outcomes. Little information is available to assist in the choice process either prior to or subsequent to the decision.

One particular version of the anarchical model is the 'garbage can model' proposed by Cohen, March and Olsen (1972) which depicts the random nature of choice in anarchical environments. (Their study was largely conducted in university settings.) They view organisational choice processes in terms of a large garbage can into which problems and solutions are dumped by organisational participants. A choice is a random event in which a solution happens to match up with a problem. Frequently solutions are present before problems are observed. When a particular problem arises someone recalls that a solution to that problem has been sitting around. For example, the data processing department of an organisation has some slack present in that one particular programmer hasn't quite enough work to keep him fully occupied. The manager of the department happens to have lunch one day with the accountant who complains that her department is slightly overloaded, but that there is not enough work to justify engaging the services of another employee. The data processing manager, recalling that his under-utilised programmer has some accounting training, offers the accountant the programmer's services for 2 days per week. Thus the chance meeting between the two managers, one with a problem and the other with a solution, leads fortuitously to the decision to deploy the programmer across the two departments.

One of the distinguishing features of the anarchical model is the part played by chance. Choices often occur as a result of chance events whereby a participant stumbles onto a solution by accident. An outcome is seen as an unintended product of certain processes. Events happen and they are then described in a systematic fashion as decisions.

### Which model?

The alternative models discussed above are representative of the diverse attempts to capture the essential nature of choice processes. Although many authors have been 'concerned with the advocacy of a particular model of choice there is increasing acknowledgement of a range of alternative models' (Baxter and Hirst, 1985). No one model can be completely descriptive of the way in which all organisations make choices, or of even a single organisation's choice processes. Features of all these models can be observed across any organisation, or even in any single decision. All organisations, for example, employ standard operating procedures, or rules. Some decisions can be observed to be the result of satisficing approaches. Most organisations have powerful individuals who attempt to have their advocated policies adopted. Chance, too, tends to play its part in decision making at different times.

## From theory to practice

Which decision model do you think best describes the way decisions are made in the organisation in which you work?

Thompson and Tuden (1959) have suggested a situational model (Figure 1-2) for considering how decisions are made. They distinguished between uncertainty (or disagreement) over objectives and uncertainty over patterns of causation which determine the consequences of action.

| | | Uncertainty of Objectives | |
|---|---|---|---|
| | | *Low* | *High* |
| **Uncertainty of Cause and Effect** | *Low* | Decisions by Computation | Decisions by Compromise |
| | *High* | Decisions by Judgment | Decision by Inspiration |

**Figure 1-2**

The following discussion attempts to link the four concepts illustrated in Figure 1-2 to particular models of choice.

- When objectives are clear and undisputed and consequences of action are known decision making is a technical matter and can be resolved by computation. In such a setting it is likely that the (bounded) rational model would be appropriate.

- As cause and effect relationships become more uncertain, but preferences are clear, there is a need to use judgment. Because the organisational procedures or SOPS model emphasises uncertainty avoidance it is probably the appropriate model in this cell.

- When consequences of action are clear but there is disagreement or uncertainty over objectives there is likely to be considerable debate and bargaining which will eventually lead to a decision by compromise. Such a situation is indicative of the political model.

- As patterns of causation become unclear with uncertainty or disagreement over objectives decision making tends to be inspirational, with rationales for action emerging in the course of the choice process. Whilst retaining some similarities to the political model, perhaps this cell might most closely align with the anarchical model.

Although the situational variables proposed by Thompson and Tuden present a useful scheme for analysing decisions the variables are probably not sensitive enough to uniquely identify a particular choice model with each cell (Baxter and Hirst, 1985), as has been attempted above. Nevertheless it is an interesting exercise to attempt to use their model as a basis for thinking about situations in which a particular model of choice may likely be evident.

It has been suggested by Eisenhardt and Zbaracki (1992) that strategic decision making is best described by the bounded rationality and political models of choice - that it is a combination of both. They say that strategic decision makers have cognitive limitations and cycle 'among rational decision making steps'. They also engage in politics. Even though chance plays a role in choice Eisenhardt and Zbaracki suggest that the garbage can model is less relevant for strategic decision making; the evidence supporting it is very limited.

Subsequent to Eisenhardt and Zbaracki's article, other researchers concerned with how decisions are made in practice in organisations have found support for the contention that strategic decisions often reflect a mix of both the bounded rational and political models of choice. For example, Dean and Sharfman (1993b, 1996) have found that in strategic decision making, political behaviour of individuals and groups within organisations and procedural (i.e. bounded) rationality can and generally do co-exist and represent two different and independent dimensions of decision making. Their findings stress that decisions are either bounded rational **and** political **or** neither.

## The case for rational analysis

Although the rational model is an ideal and not attainable because of the time, effort and costs involved and because of human cognitive limitations, it is, at least in the bounded rationality version, a basis for thinking about problems and their solutions. The very notion that one should list alternative actions, think about the possible states which may occur, and try to evaluate the consequences of each action under the identified states, provides a framework for analysis and must assist a decision maker in viewing a problem in some organised way.

Because intelligent choice is seen to be a symbol of competence in modern western civilisation (Feldman and March, 1981) individuals and organisations like to be seen to be rational. So, even in a situation where the political model is evident, the various powerful participants may advance a form of rational analysis (albeit restricted to enhance their view) to support their advocacy and as an aid to persuasion and bargaining. Standard procedures and programs may be developed in the first place from some form of rational analysis. Post-decision rationalisations observed in the anarchical model may be couched in rational terms. So despite the criticisms of rational analysis it is seen as a pervading force in choice processes in some form or other and hence there is strong justification for studying and learning rational techniques.

Researchers of decision making (Frederickson 1984, Miller 1987, Dean & Sharfman 1993a) have suggested that increased environmental turbulence, increased complexity and the existence of external control often cause organisations to move from rational to more bounded rational approaches to decision making. Whatever choice model organisations might actually use, in circumstances where environments are turbulent and operations, environments or decisions are complex, the more rational that decision makers can be the more likely it is that 'bad' decisions will be avoided.

**Study task**

Do you understand the different decision models discussed in the reading above? Try the following Self-test questions to see if you do, and then check your answers:

- Matching question 9-10
- Completion statement questions 6-10
- Multiple choice questions 6-10

## Summary

Most of the management accounting techniques commonly in use until the 1980s were developed during the period 1825-1925. The one significant development between 1925 and 1985 was in the area of capital budgeting with the introduction of discounted cash flow techniques. By 1985 the business environment had become more competitive requiring management accounting to provide new types of information for competitive support.

Since the 1970s we have seen the gradual emergence of the global economy. Australian businesses have had to become more internationally competitive or go under. To survive involves gaining and maintaining a competitive advantage. This involves strategic management decisions supported by strategic management accounting information.

The making of successful strategic decisions requires understanding decision making skills and techniques, and the input information to those decisions. How decisions are made depends on the choice processes in organisations.

A number of choice models have been examined: the rational model, the satisficing or bounded rationality model, the organisational procedures (or SOPS) model, the political model, disjointed incrementalism and the anarchical model. Features of all these models tend to be observe in any one organisation. Research seems to indicate that strategic decisions reflect the use of bounded rationality together with the political model.

**Review task**

Check your ability to use the rational model to solve a problem by attempting question 1-17 in the Question Bank.

## Self-test questions

### I Matching

You are required to match the numbered term with the letter of the most appropriate description.

| | | | | |
|---|---|---|---|---|
| 1. | _____ | Post-industrial society | A | The rational model of choice. |
| 2. | _____ | Strategic management accounting | B | Alternative strategies for solving a problem. |
| 3. | _____ | Provides information | C | Events which may influence the returns expected from pursuing a particular action. |
| 4. | _____ | Normative model | D | A model of choice in which a decision maker settles for the first solution which is found to be good enough. |
| 5. | _____ | Mean | E | A model of choice in which goals are formulated after a choice is made in order to explain or justify that choice. |
| 6. | _____ | States | F | The chance of an event occurring. |
| 7. | _____ | Probability | G | Provides information to assist managers in developing, evaluating and monitoring competitive strategies. |
| 8. | _____ | Actions | H | Period characterised by relative decline in employment in manufacturing but expansion in service sector employment. |
| 9. | _____ | Anarchical model | I | Management accounting. |
| 10. | _____ | Satisficing model | J | The expected value of a probability distribution. |

## II Completion statements

Complete each of the following statements with the most appropriate word or words.

1. There are two major flaws in the value-added approach to management accounting. It starts too late with purchases, missing opportunities for links with suppliers. It stops too early because it misses opportunities for ________________________.

2 .The objective of strategic management is to ________________________ ____________________________________________________________

3. Choice is the end product of ______________________________________

4. Simon identified three principal phases in decision making, _______________, ____________________, and ___________________.

5. Descriptive models have been constructed to ________________________ ____________________________________________________________

6. SOPS is another term used to describe the __________________________ ____________________________________________________________

7. Conflict is regarded as normal in the ______________________ model of choice.

8. Policy making is remedial, involving small adjustments at the margin in this model of choice: _____________________________.

9. Because of _________________________________ humans do not have the capacity to solve problems rationally.

10. When there is clear agreement over objectives but high uncertainty over cause and effect, decisions are made by _____________________ according to Thompson and Tuden.

## III Multiple choice

For each of the following questions, identify the correct alternative.

1. The objective of strategic management is to:

   A make strategic decisions
   B develop a financial model
   C attain and sustain superior financial performance
   D manage an organisation's strategies
   E make strategic choices

2. The value-added approach of management accounting is

   A the preferred approach in strategic management accounting
   B too internally focused
   C broader than conventional management accounting
   D external to the firm
   E related to the whole value chain

3. Strategic management accounting

   A has an internal focus
   B has an external focus
   C has neither an internal nor an external focus
   D starts from payments to suppliers and ends with receipts from customers
   E is more customer focused than conventional management accounting

**Use the following information to answer Questions 4 and 5:**

A decision maker is faced with the following conditional profits matrix:

| | **$s_1$: High Demand** | **$S_2$: Low Demand** |
|---|---|---|
| **$a_1$: Buy Machine A** | $10,000 | $2,000 |
| **$a_2$: Buy Machine B** | $8,000 | $5,000 |

Her best estimate is that the probability of a high demand is 0.4.

4. The probability of a low demand is

   A 0.4
   B 0.6
   C 0.5
   D Cannot be calculated
   E none of the above

5. The decision maker should

   A purchase Machine A because the expected value of $a_1$ is $1000 more than the expected value of $a_2$.
   B purchase Machine B because the expected value of $a_2$ is $1000 less than the expected value of $a_1$.
   C purchase neither machine because they both have negative expected values.
   D purchase either machine because they are equally desirable.
   E None of the above is correct.

**Answers to Questions 6 to 10 are to be selected from the following models of choice. Each may be used more than once.**

- A Rational
- B Satisficing
- C Organisational procedures
- D Political
- E Anarchical

6. Involves conflict, and decisions result from the exercise of persuasiveness and power.

7. Is supposed to be representative of how decisions should be made.

8. Decision making is based on the use of rules and standard operating procedures.

9. Chance plays a very large part in how decisions are made.

10. Is based on the concept of economic man.

## Self-test solutions

### I Matching

1. H 2. G 3. I 4. A 5. J 6. C 7. F 8. B 9. E 10. D

### II Completion statements

1. links with customers
2. attain and sustain superior financial performance
3. the decision making process
4. intelligence, design, choice
5. attempt to simulate or describe actual behaviour
6. organisational procedures model of choice
7. political
8. disjointed incrementalism (or muddling through)
9. cognitive limitations
10. judgment

### III Multiple choice

1. C
2. B
3. E
4. B (1-0.4 = 0.6)
5. E (Actually $E(a_1)$ = $5200 $E(a_2)$ = $6200, so should purchase B - $1000 **more**)
6. D
7. A
8. C
9. E
10. A

### Review task

Optimum weekly production = A4 - 13 cases.

| | 0.1 | 0.1 | 0.1 | 0.4 | 0.3 |
|---|---|---|---|---|---|
| **Demand:** | **S1: 10** | **S2: 11** | **S3: 12** | **S4: 13** | **S5: 14** |
| **Produce** | | | | | |
| **A1: 10** | $300 | $300 | $300 | $300 | $300 |
| **A2: 11** | $250 | $330 | $330 | $330 | $330 |
| **A3: 12** | $200 | $280 | $360 | $360 | $360 |
| **A4: 13** | $150 | $230 | $310 | $390 | $390 |
| **A5: 14** | $100 | $180 | $260 | $340 | $420 |

| | |
|---|---|
| E(A1) = $300 | 300 |
| E(A2) = $250(0.1)+$330(0.9) = | 322 |
| E(A3) = $200(0.1)+$280(0.1)+$360(0.8) = | 336 |
| E(A4) = $150(0.1)+$230(0.1)+$310(0.1)+$390(0.7) = | 342 * |
| E(A5) = $100(0.1)+$180(0.1)+$260(0.1)+$340(0.4)+$420(0.3) = | 316 |

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# Topic 2 Probability concepts and distributions

## Essential reading

**Textbook** Chapter 2, pp. 41-58.

## Objectives

At the end of this topic, you should be able to:

- demonstrate an understanding of the nature of probability and its basic foundations;
- execute simple probability calculations;
- describe and provide examples of both discrete and continuous random variables;
- convert discrete probability distributions to cumulative distributions;
- calculate the mean and standard deviation of discrete probability distributions.

## Commentary

In Topic 2, we examine some basic probability concepts and review the idea of discrete and continuous probability distributions. For some of you, this may simply be a refresher course in statistics. Others may find it completely new and will need to study the material carefully, as it provides a basis for the next few topics and the rest of the subject.

### Definition of terms

The idea of risk and probability is part of our everyday lives:

> *There is a 30% change of rain in Sydney today.*
>
> *Australia are favoured 2 to 1 over England in the next cricket test.*
>
> *There is a 50/50 chance that there will be a current account deficit this month.*

A **probability** is a statement about the likelihood that an event will occur. The probability, p, of any event or state occurring is greater than or equal to 0, and less than or equal to 1. That is,

$$0 \leq p \leq 1. \text{ (where p = probability)}$$

A probability of 0 means that an event will never occur. A probability of 1 means that an event is certain to occur.

The sum of the probabilities for all possible outcomes of an activity must equal 1. For example, if we toss a coin once, it must come down heads or tails. If it is a fair coin, there should be an equal probability of a head or a tail; that is,

p(head) = 0.5, and
p(tail) = 0.5

The sum of these two probabilities is 1.

In a pack of 52 playing cards there are four suits, each of 13 cards. The probability of drawing a Spade from a deck of playing cards is:

$$\text{p(Spade)} = \frac{13}{52} \quad \frac{\text{(number of chances of drawing spade)}}{\text{(number of total possible outcomes)}}$$

$$= \frac{1}{4} = 0.25$$

Similarly, p(Heart) = 0.25; p(Club) = 0.25; and p(Diamond) = 0.25. The sum of these four probabilities is 1, because these are the total possible outcomes.

Two or more events are said to be **mutually exclusive** if they cannot occur simultaneously. You cannot get a head and a tail in the one toss of the coin, so the two are mutually exclusive.

**Statistical independence** means that the occurrence of a first event has no effect on the probability of the occurrence of a second event. The fact that the last toss of a fair coin was heads has no influence on the probability of a head on the next toss of the coin (despite some gamblers' beliefs to the contrary).

**Statistical dependence** means that the occurrence of on event affects the probability of the occurrence of a second event. If most production workers in an organisation are male, the fact that an employee selected at random is female affects the probability that the employee is a production worker; i.e., gender and type of work are statistically dependent.

### Study task

Now check your understanding of these terms by attempting the following Self-test questions and then check your answers:

- Matching questions 1-4
- Completion statement questions 1-2
- Multiple choice questions 1-2

## Probability calculations

Refer to the following table relating to the employees of an organisation, ABC Company Ltd:

| | Gender | | |
|---|---|---|---|
| **Age** | **Male ($B_1$)** | **Female ($B_2$)** | **Total** |
| ($A_1$) Under 34 | 2100 | 900 | 3000 |
| ($A_2$) 34 – 54 | 4200 | 1800 | 6000 |
| ($A_3$) 55 or more | 700 | 300 | 1000 |
| Total | 7000 | 3000 | 10000 |

**Marginal probability**

The probability that an employee in ABC Co is a male is

$$\frac{7000}{10000} = 0.7\left[p\left(B_1\right) = 0.7\right]$$

The probability that an employee is aged between 34 and 54 is:

$$\frac{6000}{10000} = 0.6\left[p\left(A_2\right) = 0.6\right]$$

These are called marginal probabilities because we are referring to numbers around the margins – the total row, and the total column.

**Joint probabilities**

The probability that an employee selected at random is *under 34 and male* is a joint probability because it specified two characteristics:

$$\begin{aligned} &p\left(A_1 \text{ and } B_1\right) \text{or } p\left(A_1B_1\right) \\ &= p\left(A_1\right)p\left(B_1 \middle| A_1\right) \\ &= \frac{3000}{10,000} \times \frac{2100}{3000} = 0.3 \times 0.7 \\ &\qquad\qquad\qquad = 0.21 \end{aligned}$$

We can see this result directly from the table:

$$\frac{2100}{10000} = 0.21$$

**Conditional probability**

Given that an employee selected at random is a male, what is the probability that the person is under 34 years of age? We refer to the sub-population of 7000 males:

$$p(A_1|B_1) = \frac{2100}{7000} = 0.3$$

$p(A_1|B_1)$ reads: the probability of $A_1$ given $B_1$.

**Addition theorem**

What is the probability that an employee selected at random is *either* under 34 years of age *or* male?

$$\begin{aligned} p(A_1 \text{ or } B_1) &= p(A_1) + p(B_1) - p(A_1B_1) \\ &= 0.3 + 0.7 - 0.21 \\ &= 0.79 \end{aligned}$$

We subtract $p(A_1B_1)$ to avoid double counting for people who are *both* male *and* under 34. That is, the two classes are not *mutually exclusive.*

**Multiplication theorem**

Any joint probability, e.g. $p(A_1B_1)$, can be found from the multiplication theorem:

$$\begin{aligned} p(A_1B_1) &= p(A_1)p(B_1 | A_1) \\ &= (0.3)(0.7) \\ &= 0.21 \end{aligned}$$

### Statistical independence

Two variables are statistically independent if *all* conditional probabilities are equal to the corresponding marginal probabilities. That is, if

$$p(A_1 \mid B_1) = p(A_1)$$
$$p(A_1 \mid B_2) = p(A_1)$$
$$.$$
$$.$$
$$.$$
$$p(B_1 \mid A_1) = p(B_1)$$
etc.

> **Study task**
>
> Now check your understanding of probability calculations by attempting the following Self-test questions and then check your answers:
>
> - Matching questions 5-6
> - Completion statement questions 3-4
> - Multiple choice questions 3-6

## Populations and samples

A **population** is a complete set or collection of objects of interest. It is the aggregate or totality of items under consideration in connection with a specific problem, and is called the population under study. If we wanted to study the age distribution of students enrolled in the Master of Accountancy at Charles Sturt University, then all the students enrolled in the Master of Accountancy in Australia and overseas constitute the population of interest. If, however, we were interested only in the ages of the female students, then the population would be all the female students enrolled in the Master of Accountancy. These populations are of limited size, and are called **finite** populations. In contrast, some populations are indefinitely large, and are called **infinite** populations, e.g. the number of grains of sand on a beach, or the number of drug tablets manufactured in a production process.

If we wish to establish some facts about every person or unit in a **finite population** we take a census - that is, we ask every person, or measure some attribute of every unit. A well known example is the census conducted every 5 years into the Australian population, in which every person is accounted for and provides the information requested. Of course, it is not possible to take a census for **infinite populations**.

Rather than take a census, decision makers frequently base decisions on a **sample** from the population. They take a **random** sample (such that every member of the population should have an equal chance of being selected), measure the property of interest for each member in the sample, and then draw conclusions about the population in general, from the average or mean results for the sample. Thus, the conclusions that are drawn are influenced by the representativeness of the sample. If the sample is biased at all, then incorrect inferences would be made about the population.

Even random samples are not necessarily representative of the population. For example, if we were interested in the average age of students enrolled in the Master of Accountancy, we could take a random sample of students, find their ages, and calculate the mean age for the sample. Now it is possible that the sample, by chance, picked up all the oldest students, or all the youngest. If so, we would draw the wrong conclusions about the population.

We can guard against such biases by selecting a **large**, random sample. The smaller the sample, the less the sample average is representative of the population. As the sample size increases, the sample average tends to be closer to the population average. It is actually surprising that, in general, a sample does not have to be very large to obtain reliable results. Sometimes samples as small as 4 can give quite accurate information about the population especially when sampling from production processes. In general we would prefer samples of at least 30, if that is possible. There are techniques to calculate the minimum sample size needed to draw valid conclusions, but these are based on trading off different types of statistical error and are beyond the scope of this subject.

## Random variables

A variable that can vary randomly over a defined range of values is called a random variable. For example, the number of cars sold daily (call it x) by a car dealer may vary from 0 to 10. If you were to select any particular day at random, then x will take a definite value for that day, say 4. If you repeat the experiment many times, the value of x will vary across the full range 0 to 10.

Random variables may be **discrete** or **continuous.** A discrete random variable can assume only a finite or limited set of whole values. Discrete variables may have numerical values, represented by distinct integers 0, 1, 2 … Discrete random variables may also be **categorical.** For example, responses to a questionnaire may be: Strongly Agree, Agree, Neutral, Disagree, Strongly Disagree. This categorical random variable can be represented by integers, using numbers 1, 2, 3, 4, 5 for the categories, e.g. 1 = Strongly agree, 2 = Agree, etc.

A continuous random variable has an infinite set of values, grading imperceptibly into each other. Time, length and mass are examples of continuous variables, e.g. how long a light bulb burns before it expires. We could measure time in days, or hours, or minutes, or seconds, or hundredths of a second. No matter how fine a scale we use, there can be a finer one still. Time is, in theory, infinitely divisible.

## Probability distributions

The set of all possible values of a random variable and their associated probabilities is referred to as a **probability distribution.**

## Discrete random variables

When we have a discrete random variable, there is a probability assigned to each possible value, and the probabilities must sum to unity (i.e. sum to one).

Suppose a car dealer sells between 0 and 5 cars daily. The 'daily car sales' is a discrete random variable. It is discrete because it can only take on integer values. Let us label this variable **x**. If we know the **relative frequency** of daily sales, we have a measure of probability. Suppose that over the last 100 days the car dealer has the following sales results, shown in columns 1 and 2:

| *Sales Units (x)* | *Number of Days* | *Probability [p(x)]* |
|---|---|---|
| 0 | 10 | 0.10 |
| 1 | 25 | 0.25 |
| 2 | 30 | 0.30 |
| 3 | 20 | 0.20 |
| 4 | 10 | 0.10 |
| 5 | 5 | 0.05 |
| | 100 | 1.00 |

Using the relative frequency approach, 0 cars were sold on 10 days out of 100, which gives a probability of 10/100 = 0.1, column 3. One car was sold on 25 days out of 100, giving a probability of 25/100 = 0.25. Similarly the other probabilities were calculated.

The **mean** or **average** value of a discrete probability distribution is represented by its **expected value.** The expected value, **E(x)**, is found by multiplying each possible value by its probability, and summing. Thus the mean daily sales, or expected daily sales, is calculated as:

$\Sigma(x)p(x)$ [where p(x) = probability of x]
$= 0(0.10) + 1(0.25) + 2(0.30) + 3(0.20) + 4(0.10) + 5(0.05)$
$= 2.1$ cars

This means that the average daily sales are between 2 and 3 cars, but closer to 2.

**From theory to practice**

This calculation could be performed using a spreadsheet. Try this out and compare your work with the discussion below.

The fourth column (D) shows the results of multiplying the first column (A) by the third column (C), i.e. **(x)p(x)**. The average daily sales is the total of column D, i.e. **Σ(x)p(x)** = 2.1:

| *Sales Units (x)* | *Number of days* | *Probability p(x)* | *Expected sales (x)p(x)* |
|---|---|---|---|
| 0 | 10 | 0.10 | 0.00 |
| 1 | 25 | 0.25 | 0.25 |
| 2 | 30 | 0.30 | 0.60 |
| 3 | 20 | 0.20 | 0.60 |
| 4 | 10 | 0.10 | 0.40 |
| 5 | 5 | 0.05 | 0.25 |
| | 100 | E(x)= | 2.10 |

The mean of a distribution is a measure of **central tendency.** People are also interested in the variability of a distribution - its dispersion, or how spread out it is.

One measure of dispersion is the **range,** the difference between the largest and smallest value. In this case the range would be 5-0 = 5 cars.

A more useful measure of dispersion is the **variance,** which is found by calculating the difference between each value and the mean, squaring it, and multiplying it by the probability of that value, and summing all the squared figures. The variance, $\boldsymbol{\sigma_x^2}$, of daily car sales is calculated as follows:

$$\begin{aligned}\sigma_x^2 &= \Sigma[x - E(x)]^2\, p(x) \\ &= (0-2.1)^2(0.10) + (1-2.1)^2(0.25) + (2-2.1)^2(0.30) \\ &\quad + (3-2.1)^2(0.20) + (4-2.1)^2(0.10) + (5-2.1)^2(0.05) \\ &= 1.69\end{aligned}$$

### From theory to practice

This calculation could of course be performed using a spreadsheet. Again, try this out and see whether you get the same result as shown below.

The total of the last column is the variance:

| Sales units (x) | Number of days | Probability p(x) | | Expected sales (x)p(x) | x-E(x) | $[x-E(x)]^2$ | $[x-E(x)]^2p(x)$ |
|---|---|---|---|---|---|---|---|
| 0 | 10 | 0.10 | | 0.00 | -2.10 | 4.41 | 0.4410 |
| 1 | 25 | 0.25 | | 0.25 | -1.10 | 1.21 | 0.3025 |
| 2 | 30 | 0.30 | | 0.60 | -0.10 | 0.01 | 0.0030 |
| 3 | 20 | 0.20 | | 0.60 | 0.90 | 0.81 | 0.1620 |
| 4 | 10 | 0.10 | | 0.40 | 1.90 | 3.61 | 0.3610 |
| 5 | 5 | 0.05 | | 0.25 | 2.90 | 8.41 | 0.4205 |
| | 100 | 1.00 | E(x)= | 2.10 | | | 1.6900 |

A related measure is the **standard deviation,** $\sigma_x$, which is, the square root of the variance:

$$\sigma_x = \sqrt{\sigma_x^2} = \sqrt{1.69} = 1.3$$

### Continuous random variables

Continuous random variables can take on an infinite number of values. As with discrete probability distributions, the sum of the probabilities must equal one. The distribution is described by a continuous mathematical function, called the **probability density function, f(x)**.

We cannot meaningfully talk about the probability of a particular value, but we must refer to the probability of a range of values. For example, a machined part could vary from 5.06 to 5.30 grams, with 5.18 grams being the expected value. The probability that its mass is between 5.22 and 5.26 grams is the shaded area under the curve in Figure 2-1.

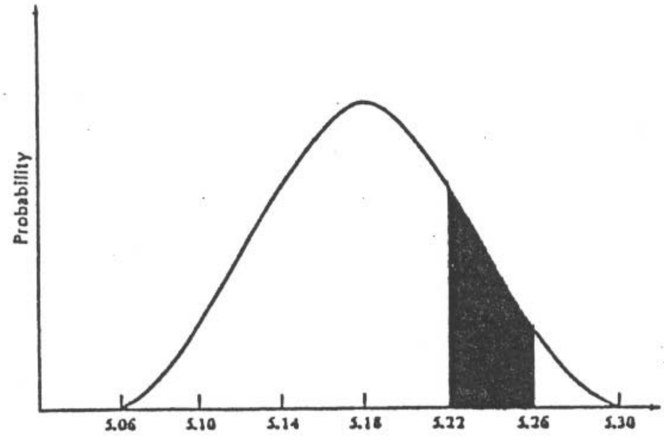



**Figure 2.1**

This probability can be calculated using integral calculus, but in our work we will use particular distributions whose areas are tabulated for us (e.g. the normal distribution – see Topic 4). Note that, since the sum of the probabilities must equal 1, the total area under a curve representing a probability density function must also equal 1.

## Cumulative distributions

In some situations we desire information in a form which is a modification of the probability distributions already discussed. Let us look again at the distribution of daily sales for the car dealer, reproduced below:

| *Sales Units (x)* | *Probability* |
|---|---|
| 0 | 0.10 |
| 1 | 0.25 |
| 2 | 0.30 |
| 3 | 0.20 |
| 4 | 0.10 |
| 5 | 0.05 |
| | 1.00 |

**'Less Than' distribution**

For example, our interest which might centre on the probability of selling a given number of cars, or less, on any one day. We find this by developing the cumulative probability distribution:

| *Sales Units (x)* | *Probability* | *Cumulative Probability* |
|---|---|---|
| 0 | 0.10 | 0.10 |
| 1 | 0.25 | 0.35 |
| 2 | 0.30 | 0.65 |
| 3 | 0.20 | 0.85 |
| 4 | 0.10 | 0.95 |
| 5 | 0.05 | 1.00 |
| | 1.00 | |

The probability of selling 1 car *or less* is equal to the sum of the probabilities of selling 0 or 1 car:

0.10 + 0.25 = 0.35

To find the probability of selling 2 cars or less, we add to 0.35 the probability of selling 2 cars:

0.35 + 0.30 = 0.65.

Then add the probability of selling 3 cars (0.20) to 0.65 to get the probability of selling 3 cars or less, and so on.

Thus, on 85% of days the dealer will sell 3 cars or less (cumulative probability for 3 = 0.85). This same answer could have been derived, of course, by looking at the probability distribution (column 2) and summing the first four probabilities.

**'More Than' distribution**

The cumulative distribution could also have been computed to indicate the probability of selling a given number of cars, *or more*, on any day. To show this 'more than' distribution we would set up the cumulative distribution as follows:

| *Sales Units (x)* | *Probability* | *Cumulative Probability* |
|---|---|---|
| 0 | 0.10 | 1.00 |
| 1 | 0.25 | 0.90 |
| 2 | 0.30 | 0.65 |
| 3 | 0.20 | 0.35 |
| 4 | 0.10 | 0.15 |
| 5 | 0.05 | 0.05 |
| | 1.00 | |

Thus we start with 1.00 for the lowest value, daily sales of 0, because there is a probability of 1 (it is a certainty) that sales will be 0 or more. For the probability of sales of 1 or more we subtract from 1.00 the probability of sales of 0 cars:

$$1.00 - 0.10 = 0.90.$$

The probability of sales of 2 cars or more is found by subtracting from 0.90 the probability of sales of 1 car:

$$0.90 - 0.25 = 0.65.$$

Alternatively, the same answer could be obtained by summing the probability of sales of 2, 3, 4 and 5:

$$0.30 + 0.20 + 0.10 + 0.05 = 0.65.$$

In a similar manner the distribution is completed, as shown above.

**Study task**

Now check your understanding of the above section by attempting the following Self-test questions and then check your answers:

- Matching questions 7-10
- Completion statement questions 5-10
- Multiple choice questions 7-10

## Summary

The term probability refers to the likelihood that an event will occur. The probability, p, of any event occurring lies between 0 and 1. The sum of the probabilities for all possible outcomes of an activity must equal 1. Two events are said to be mutually exclusive if they cannot occur simultaneously. Two or more events can be either statistically dependent or independent.

Three main types of probability calculations are marginal probability, joint probability and conditional probability. Two major theorems relating to probability calculations are the addition theorem and the multiplication theorem.

A population represents the totality of items under consideration. Populations can be either finite or infinite. Decision makers frequently base their decisions on a random sample taken from the population under consideration.

A random variable may be discrete or continuous. A probability distribution is the set of all possible values of a random variable and their associated probabilities. The mean of a probability distribution gives the average value, while the variance or standard deviation refers to its dispersion.

A probability distribution can be converted to a cumulative probability distribution. We can convert discrete distributions into two types of cumulative distributions, a 'less than' distribution or a 'more than' distribution. You will learn later that each type can be useful.

## Self-test questions

### I Matching

You are required to match the numbered term with the letter of the most appropriate description.

1. _____ Probability
2. _____ Mutually exclusive
3. _____ Statistical independence
4. _____ Statistically dependent events
5. _____ Joint probability
6. _____ Conditional probability
7. _____ Population
8. _____ Finite population
9. _____ Probability distribution
10. _____ Mean

A The probability of a first event occurring has no effect on the probability of the second event occurring.

B The totality of items under consideration in connection with a specific problem.

C The set of all possible values of a random variable and their associated probabilities.

D A population of limited size.

E The probability of an event occurring given that another event has taken place.

F The chance of an event occurring.

G The occurrence of one event affects the probability of occurrence of a second event.

H The probability of two independent events occurring.

I Only one event can occur on any one trial.

J The expected value of a probability distribution.

## II Completion statements

Complete each of the following statements with the most appropriate word or words.

1. Two events which cannot occur simultaneously are said to be _____________.

2. If a set of events is mutually exclusive and collectively exhaustive this implies that ________________________________.

3. The probability of two or more independent events occurring is the _________________________________.

4. Given two statistically independent events (A,B), the conditional probability of p(A|B) = ___________________.

5. Time, length and mass are examples of ________________________________.

6. The number of people attending a rock concert is a ______________________ random variable.

7. To establish some facts about every member of a population we take a ___________________.

8. A random sample ensures that every member of a population has _________________________________.

9. The mean of a distribution is a measure of _______________________________.

10. The standard deviation of a distribution is a measure of ____________________.

## III Multiple choice

For each of the following questions, identify the correct alternative.

1. The sum of the probabilities for all possible outcomes of an activity must:

   A not exceed one
   B equal one
   C exceed one
   D be equal to or greater than zero, or less than or equal to one
   E equal to zero

2. Which of the following pairs of events is independent?

   A snow in Victoria and election results in New South Wales
   B St George wins the minor premiership and St George wins the grand final
   C your education and your income
   D Government taxes on petrol and the price of petrol
   E none of the above

3. A conditional probability p(B|A) is equal to its marginal probability p(B) if:

   A it is a joint probability
   B statistical dependence exists
   C statistical independence exists
   D the events are mutually exclusive
   E p(A) = p(B)

**Questions 4, 5 and 6 are based on the table of employees in ABC Co. Ltd.**

| | Gender | | |
|---|---|---|---|
| **Age** | **Male ($B_1$)** | **Female ($B_2$)** | **Total** |
| ($A_1$) Under 34 | 2100 | 900 | 3000 |
| ($A_2$) 34 – 54 | 4200 | 1800 | 6000 |
| ($A_3$) 55 or more | 700 | 300 | 1000 |
| Total | 7000 | 3000 | 10000 |

4. The joint probability that an employee selected at random is female and between 34 and 54 years of age is:

   A 0.03
   B 0.18
   C 0.30
   D 0.60
   E none of the above

5. Given that an employee selected at random is aged 40, what is the probability that the person is female?

   A 0.10
   B 0.18
   C 0.30
   D 0.42
   E 0.60

6. What is the probability that an employee selected at random is either female or 55 or over?

   A 0.03
   B 0.10
   C 0.23
   D 0.30
   E 0.37

7. A continuous random variable:

   A has values which are integers only
   B has no specific value at all
   C has the same values continuously over time
   D cannot be categorical
   E cannot have an expected value

8. After evaluating the responses to a question in a survey, the researcher constructed the following table:

| Response | Random variable, x | Probability |
|---|---|---|
| Strongly agree | 5 | 0.10 |
| Agree | 4 | 0.15 |
| Neutral | 3 | 0.40 |
| Disagree | 2 | 0.20 |
| Strongly disagree | 1 | 0.15 |

   This table describes:

   A a standard normal variable
   B discrete probability distribution
   C a continuous probability distribution
   D a cumulative distribution
   E a statistical process.

**Use the following information to answer questions 9 and 10:**

| Daily sales of cars | Probability |
|---|---|
| 1 | 0.2 |
| 2 | 0.4 |
| 3 | 0.2 |
| 4 | 0.1 |
| 5 | 0.1 |

9. The average daily sales of cars is:

   A 3.0
   B 2.1
   C 0.5
   D 2.5
   E none of the above

10. The standard deviation of daily car sales is (to 2 decimal places):

    A 1.45
    B 2.50
    C 1.20
    D 1.58
    E none of the above

## IV Exercises

1. A lot of 10,000 parts, produced on four machines, was graded according to three grades. These results were:

| | | Machine | | | | |
|---|---|---|---|---|---|---|
| | **Grade** | **W** $B_1$ | **X** $B_2$ | **Y** $B_3$ | **Z** $B_4$ | **All Machines** |
| $A_1$ | **Satisfactory** | 3,200 | 800 | 2,400 | 1,600 | 8,000 |
| $A_2$ | **Rework** | 600 | 150 | 450 | 300 | 1,500 |
| $A_3$ | **Scrap** | 200 | 50 | 150 | 100 | 500 |
| | **All Grades** | 4,000 | 1,000 | 3,000 | 2,000 | 10,000 |

a. One of the parts is to be selected at random. What is the probability that it was produced by Machine W and should be reworked?

b. What is the symbolic notation for the probability that a part selected at random

   i. was produced by Machine X?
   ii. was produced by Machine Z and is not satisfactory?
   iii. was produced by Machine Y and should be scrapped
   iv. needs to be reworked?
   v. needs to be scrapped, given that it was produced by Machine W?

c. What is each of the probabilities in (b)?

d. What is each of the following probabilities?

   i. $p(A_2 \text{ and } B_3)$
   ii. $p(A_1)$
   iii. $p(B_2 | A_2)$
   iv. $p(A_1 B_4)$
   v. $p(B_3)$

e. Are the two variables (machine and grade) statistically independent in the population of 10,000 parts? Explain how you arrive at your answer?

2. Suppose the daily demand for pies from a take-away food shop is represented by the following distribution:

| Daily Demand (pies) | Probability |
|---|---|
| 50 | 0.1 |
| 75 | 0.3 |
| 100 | 0.3 |
| 125 | 0.2 |
| 150 | 0.1 |
| | 1.0 |

a. Convert the distribution to a cumulative probability distribution of the 'less than' type.

b. Convert the distribution to a cumulative probability distribution of the 'more than' type.

## Self-test solutions

### I Matching

1. F 2. I 3. A 4 .G 5. H 6. E 7. B 8. D 9. C 10. J

### II Completion statements

1. mutually exclusive
2. one and only one of the events can occur on any trial
3. joint probability
4. p(A)
5. continuous variables
6. discrete
7. census
8. an equal chance of being selected
9. central tendency
10. dispersion

### III Multiple choice

1. B
2. A
3. C
4. B $p(B_2A_2) = (0.3)(0.6) = 0.18$
5. C $p(B_2|A_2) = 1800/6000 = 0.3$
6. E $p(B_2 \text{ or } A_3) = 0.3 + 0.1 - 0.03 = 0.37$
7. D
8. B
9. D $1(0.2) + 2(0.4) + 3(0.2) + 4(0.1) + 5(0.1) = 2.5$
10. C

$$\sqrt{[(1-2.5)^2(0.2)+(2-2.5)^2(0.4)+(3-2.5)^2(0.2)+(4-2.5)^2(0.1)+(5-2.5)^2(0.1)}$$

$$= \sqrt{(0.45+0.10+0.05+0.225+0.625)}$$
$$= \sqrt{1.45}$$
$$=1.20$$

### IV Exercises

1. a.

$$\begin{aligned} p(B_1 \text{ and } A_2) \text{ or } p(B_1A_2) &= p(B_1)p(A_2) \\ &= (4000/10000)(1500/10000) \\ &= (0.4)(0.15) \\ &= 0.06 \end{aligned}$$

b. i. $p(B_2)$

ii. $p(B_4A_2)+p(B_4A_3)$ or $p(B_4)-p(B_4A_1)$

iii. $p(B_3A_3)$ [or $p(B_3$ and $A_3)$]

iv. $p(A_2)$

v. $p(A_3 \mid B_1)$

c. i. 1000/10000 = 0.1

ii. 0.03 + 0.01 = 0.04 [or 0.2 – 0.16 = 0.04]

iii. 0.015 [150/10000]

iv. 0.15 [1500/10000]

v. 200/4000 = 0.05

d. i. $p(A_2 \text{ and } B_3) = 450/10000 = 0.045$

or $p(A_2 \text{ and } B_3) = p(A_2)p(B_3) = (0.15)(0.3) = 0.045$

ii. $p(A_1) = 8000/10000 = 0.8$

iii. $p(B_2 \mid A_2) = 150/1500 = 0.1$

iv. $p(A_1B_4) = p(A_1)p(B_4) = (0.8)(0.2) - 0.16$

v. $p(B_3) = 3000/10000 = 0.3$

e. Yes. For statistical independence all conditional probabilities should equal corresponding marginal probabilities, e.g.

$p(A_1 \mid B_1) = p(A_1)$
$p(A_1 \mid B_2) = p(A_1)$
$p(A_2 \mid B_1) = p(A_2)$
$p(B_1 \mid A_1) = p(B_1)$ etc.

This is the case with this example. Thus machine and grade are statistically independent, indicating that the percentage of each grade is the same regardless of which machine is used.

2. a.

| Daily Demand | Probability | Cumulative Probability | |
|---|---|---|---|
| 50 | 0.1 | 0.1 | |
| 75 | 0.3 | 0.4 | (0.1 + 0.3) |
| 100 | 0.3 | 0.7 | (0.4 + 0.3) |
| 125 | 0.2 | 0.9 | (0.7 + 0.2) |
| 150 | 0.1 | 1.0 | (0.9 + 0.1) |
| | 1.0 | | |

b.

| Daily Demand | Probability | Cumulative Probability | |
|---|---|---|---|
| 50 | 0.1 | 1.0 | |
| 75 | 0.3 | 0.9 | (1.0 – 0.1) |
| 100 | 0.3 | 0.6 | (0.9 – 0.3) |
| 125 | 0.2 | 0.3 | (0.6 – 0.3) |
| 150 | 0.1 | 0.1 | (0.3 – 0.1) |
| | 1.0 | | |

# Topic 3 Decision analysis – A rational approach

## Essential reading

**Text** Chapter 3, pp. 89-102, 110-119.

**Reading 3.1** Tversky, A., & Kahneman, D. (1977). Judgement under uncertainty: Heuristics and biases. In G. M. Kahneman & H. Thomas (Eds.), *Modern decision analyzing* (pp. 39-61). Penguin.

## Objectives

At the end of this topic, you should be able to:

- prepare a payoff matrix or decision tree for given decision problems;
- use criteria such as maximax, maximin, minimax regret and Laplace, to choose the best action;
- determine the optimum action based on maximisation of expected utility;
- use sensitivity analysis to answer 'what if' questions;
- use marginal analysis for discrete and continuous probability distributions;
- recognise the use of heuristics in decision making and be aware of the biases that can result from using heuristics.

## Commentary

In Topic 1, the idea of and the concepts behind a rational approach to decision making were introduced. In this topic we explore various techniques of rational decision making, under varying degrees of uncertainty using both a decision matrix and a decision tree approach. The marginal analysis technique is also discussed as an appropriate method in certain circumstances.

Decisions are always future oriented. People make decisions about what to do in the future, even if it is only seconds from now. Since no one can accurately predict the future (especially the distant future) there is some degree of uncertainty as to the likely outcome of any action. The motorist, for example, in selecting a particular route, may not know that there has been an accident ahead, causing a great delay. If, however, (s)he switches on the car radio and hears a traffic report, (s)he may become aware of the impending trouble. Armed with such additional information (s)he is in a position to make a better decision as to what route to take. Thus better information can improve decision making.

At any particular time, however, the decision maker (hereafter DM) has a given set of information. (S)he must choose either to make a decision based on existing information (the focus of this topic), or to seek further information prior to making the decision (examined in Topic 5). In either case, available information can never be perfect because inevitably the future is uncertain. Consequently a DM always faces some degree of uncertainty with respect to the consequences of proposed actions.

Such uncertainty is usually assumed to relate to the probability of occurrence of each identified state of nature, rather than to which states *per se* are possible or what the payoffs are likely to be under each state. Two degrees of uncertainty are considered, complete uncertainty and partial uncertainty. In either case a payoff matrix is a useful tool for analysing decisions.

## Payoff matrix

The analysis of decisions can be facilitated by presenting relevant information in the form of a matrix, referred to variously as a **payoff matrix**, a **decision matrix**, or a **conditional profits (or losses) matrix**.

Typically possible actions (or strategies, or alternatives) are shown in the rows and anticipated states of nature (or events) appear in the columns. Each cell of the matrix contains the estimated payoff (profit, loss, revenue, cost, etc) associated with the action-state pair determining that cell location.

Consider Bill Holden, a motor car manufacturer, planning next year's model. He produces only one model, and changes it each year. This DM's problem concerns what size car to manufacture next year. He determines three possible actions **($a_j$)**:

$a_1$: Make a large car
$a_2$: Make a medium-sized car
$a_3$: Make a small car

The outcome of each action will be influenced partly by what size car will sell best. The DM decides that the predominant influence on sales is likely to be the future price of petrol, and accordingly he identifies two states of nature **($s_i$)**:

$s_1$: Petrol price will decrease
$s_2$: Petrol price will increase

The combination of these three actions and two states can be shown in a 3x2 matrix, Table 3-1.

**Table 3-1:** Payoff Matrix

| Actions | States of Nature | |
|---|---|---|
| | *$s_1$: Petrol price decrease* | *$s_2$: Petrol price increase* |
| *$a_1$: Large car* | | |
| *$a_2$: Medium car* | | |
| *$a_3$: Small car* | | |

There are 3×2, or 6 possible payoffs to be estimated, one for each action-state pair, in order to complete the matrix. The DM can calculate the six conditional payoffs from the following estimates prepared by organisational executives.

| Actions | Sales Forecasts | (Units) | Unit Contribution |
|---|---|---|---|
| | *$s_1$: Price decrease* | *$s_2$: Price increase* | |
| *$a_1$: Large car* | 100 000 | 40 000 | $3000 |
| *$a_2$: Medium car* | 130 000 | 120 000 | $1500 |
| *$a_3$: Small car* | 150 000 | 200 000 | $1000 |
| Annual Fixed Costs: $170 000 000 | | | |

Using these estimates, the payoff matrix is completed as shown in Table 3-2.

**Table 3-2:** Completed Payoff Matrix (in millions of dollars)

| Actions | States of Nature | |
|---|---|---|
| | *$s_1$: Petrol price decrease* | *$s_2$: Petrol price increase* |
| *$a_1$: Large car* | $130M | -$50M |
| *$a_2$: Medium car* | $25M | $10M |
| *$a_3$: Small car* | -$20M | $30M |

For example, if he makes a large car ($a_1$) and petrol prices decrease ($s_1$) he expects to sell 100 000 cars each of which returns $3000 contribution margin (i.e. unit selling price less unit variable costs), giving a total contribution (variable profit) of $300 000 000. Deducting annual fixed costs of $170 000 000 gives an estimated annual net profit of $130 000 000. Similarly the other five payoffs are calculated. In two cases net losses would be sustained, while the other four possibilities provide positive net incomes.

Decision theory requires that the **utility** of the payoff to the DM should be entered in each cell of the payoff matrix. In our case it is assumed that utility is measured by dollar incomes; that is, the DM is **risk neutral** and has a linear utility function - more on this later.

The payoff matrix provides a concise summary of the DM's problem. Unfortunately the DM does not know which of the two states will occur. If he did (called **decision making under certainty**) the decision would be trivial; for example, if he knew that the price of petrol would decrease he would, as a rational DM, choose a large car for a return of $130M as against $25M for a medium car or a loss of $20M for a small car. Or, if he knew for sure that the price of petrol would increase he would manufacture a small car to maximise his return at $30M. But he does not know which state will prevail, and hence this is a case of **decision making under uncertainty**.

There are differing degrees of uncertainty:

1. **complete uncertainty**, where there is an absence of any knowledge of the likelihood of occurrence of the various states, and

2. **partial uncertainty**, where either objective or subjective probability estimates of the likelihood of occurrence of the states are possible.

## Complete uncertainty

Given complete uncertainty (i.e. no knowledge of the probability of occurrence of the states) the DM needs a decision rule by which to select the 'best' action. Unfortunately there is no unique decision rule available. Rather, decisions can be based on any one of a number of proposed criteria, none of which can be regarded as the **best criterion**, but each reflects a different attitudinal response to uncertainty. The particular criterion adopted by a DM depends upon personal preference and attitudes to risk. The use of different criteria can result in the selection of different actions. Four common criteria are illustrated.

### Maximax criterion

Suppose that the DM is a complete **optimist** who assumes that Nature will treat him kindly (he feels lucky). Then he might follow the **Maximax criterion** suggested by Hurcwicz (1951). That is, he will identify the maximum payoff associated with each strategy, and select the strategy with the largest maximum payoff, i.e. maximax.

From Table 3-2, by looking across each row he can select the maximum payoff for each action. For example, *Make a large car* ($a_1$) can return $130M if petrol prices decrease, or -$50M if petrol prices increase. The maximum of these is $130M. Similarly the maximum payoffs for $a_2$ and $a_3$ are observed:

| Action | Maximum Payoff |
|---|---|
| $a_1$ | $130M * |
| $a_2$ | $25M |
| $a_3$ | $30M |

He then selects the maximum of these maxima, $130M, which is starred, indicating that he should make a large car ($a_1$). Of course, if petrol prices increase he will finish up losing $50M, but he is gambling on being lucky. Note that if the payoffs are expressed as costs the equivalent criterion for an optimist is **minimin** (find the minimum cost associated with each action and select the action with the smallest minimum).

### Maximin criterion

Wald (1945) suggested that the DM should be completely pessimistic and act as if Nature would always be malevolent. The DM adopts a defensive attitude by asking himself, 'What is the worst result that I can expect from each action?' He then settles for the best of these worst outcomes; that is, he selects the strategy with the largest minimum payoff. From Table 3-2, by scanning across each row, the DM can select the minimum payoff associated with each action:

| Action | Minimum Payoff |
|---|---|
| $a_1$ | -$50M |
| $a_2$ | $10M * |
| $a_3$ | -$20M |

He then chooses the maximum of these minima, $10M, so that he should make a medium car ($a_2$). If the payoffs are expressed as costs the equivalent criterion for a pessimist is **minimax**.

### Criterion of regret (minimax regret)

Savage (1951) has suggested an opportunity loss approach. After the DM knows the outcome, he can experience regret. With hindsight, once he knows which particular state of nature occurred, he may experience regret that he had not selected a different action. Savage argues that the DM should attempt to minimise this regret which he may experience, where the amount of regret is measured by the difference between the payoff associated with a chosen action and the optimum payoff under a given state.

The key to preparing a regret matrix is to work **down** and not across the payoff matrix. In other words, begin with the assumption that a particular **state** has occurred, and compare the payoffs **down** that column of the matrix, calculating the amount of regret that each action for that particular state may produce in comparison with the optimal action for that state.

Thus, on examining Table 3-2, suppose that the price of petrol actually decreased ($s_1$). If the DM had selected $a_1$ (large car) he would experience no regret because he would have received the largest possible payoff under $s_1$. If, however, he had chosen $a_2$ (medium car) for a return of $25M, he would experience some regret that he had not had the wisdom to have selected $a_1$. The measure of this regret is the difference between the $a_2$ payoff of $25M and the maximum possible payoff of $130M, that is $130M - $25M, or $105M. Similarly, selection of $a_3$ would lead to a regret of $150M [$130M - (-$20M)].

Alternatively, suppose that the price of petrol actually increased ($s_2$). In this case selection of $a_3$ would have been the optimum choice, with no regret being experienced. Selection of $a_1$ would result in $80M regret [$30M - (-$50M)], while the choice of $a_2$ would lead to regret of $20M ($30M - $10M). These measurements can be presented in a regret matrix:

| | Regret Matrix | |
|---|---|---|
| | $s_1$ | $s_2$ |
| $a_1$ | $0 | $80M |
| $a_2$ | $105M | $20M |
| $a_3$ | $150M | $0 |

Savage then proposes a variant of the Wald criterion. The DM employs the minimax approach to the regret matrix. He looks across each row of the regret matrix to find the maximum regret for each action, and then selects the action with the minimum maximum, in this case $a_1$ (large car):

| Action | Maximum Regret | |
|---|---|---|
| $a_1$ | $80M * | (Minimax) |
| $a_2$ | $105M | |
| $a_3$ | $150M | |

**Laplace criterion**

In the absence of any special reasons for thinking that one state is more likely than another, the Laplace criterion suggests that the DM should treat each state as equally likely, and select the action with the highest expected utility. In our example, if each of the two states is equally likely, they each have a probability of 0.5 of occurring:

| | $p(s_1) = 0.5$ | $p(s_2) = 0.5$ |
|---|---|---|
| | *$s_1$: Petrol price decrease* | *$s_2$: Petrol price increase* |
| *$a_1$: Large car* | $130M | -$50M |
| *$a_2$: Medium car* | $25M | $10M |
| *$a_3$: Small car* | -$20M | $30M |

The expected utility of $a_1$, i.e. $E(U|a_1)$, = 130(0.5) - 50(0.5) = 40
$E(U|a_2) = 25(0.5) + 10(0.5) = 17.5$
$E(U|a_3) = -20(0.5) + 30(0.5) = 5$

The highest expected utility, $40M, is for $a_1$ and hence a large car is the Laplace choice.

### Study task

To determine how well you understand the concepts of decision making under uncertainty and if you can apply the four techniques covered, attempt the following Self-test questions and then check your answers:

- Matching questions 1-3
- Completion statement questions 1-3
- Multiple choice questions 1,3,4,5,6

### From theory to practice

Which of the four techniques would be your personal choice if you were faced with a decision under complete uncertainty? What does this imply about your attitude to risk?

## Partial uncertainty

Under partial uncertainty the DM can attach probabilities to the occurrence of the states of nature. This enables the DM to calculate expected values for each action. The action with the highest expected value is the rational choice.

### Objective and subjective probabilities

It was shown above that when the DM has no knowledge of the likelihood of occurrence of the states he may resort to using one of the described criteria (maximax, maximin etc) to enable him to select a strategy. The existence of complete uncertainty, however, is rare. Often there exists some guide as to the probability of occurrence of the states, either from historical records, or simply in the mind of the DM (i.e. a gut feeling).

When the probabilities are based on objective evidence such as historical records, we refer to them as **objective probabilities**. If the probabilities are simply estimated by the DM on the basis of his knowledge, experience and best guess, the term **subjective probabilities** is used. Whether the probabilities are objective or subjective is immaterial to the subsequent analysis, except that perhaps more confidence might be placed in decisions based on objective evidence.

### State probabilities and expected utility

Suppose that the motor car manufacturer believes that it is three times as likely that petrol prices will increase rather than decrease. Such a belief implies probability estimates of 0.25 for a decrease and 0.75 for an increase. These probabilities can be incorporated in the analysis, and are placed above the states in Table 3-3, where **$p(s_1)$** means 'the probability of state 1'.

**Table 3-3:** Payoff Matrix including state probabilities

| | $p(s_1) = 0.25$ | $p(s_2) = 0.75$ |
|---|---|---|
| | *$s_1$: Petrol price decrease* | *$s_2$: Petrol price increase* |
| *$a_1$: Large car* | $130M | -$50M |
| *$a_2$: Medium car* | $25M | $10M |
| *$a_3$: Small car* | -$20M | $30M |

Under the laws of probability, the sum of the probabilities across the states must equal unity. That is,

$$\sum_{i=1}^{S} p(s_i) = 1$$

where S is the set of states.

In order to determine the preferred action, it is necessary to calculate the expected utility of each of the three actions and select that action which results in the largest value. In other words, the rational DM's decision rule is to maximise his expected utility. The three calculations are as follows, where **$E(U|a_j)$** is read as 'expected utility of action $a_j$':

$E(U|a_1) = 130(0.25) - 50(0.75) = -5$
$E(U|a_2) = 25(0.25) + 10(0.75) = 13.75$
$E(U|a_3) = -20(0.25) + 30(0.75) = 17.5$ *

The DM's optimum action is $a_3$ (small car) because it has the largest expected utility, $17.5M, which is asterisked. That is, E(U|a*) = $17.5M, where a* is the optimum action.

The formal notation for expected utility calculations is,

$$E(U \mid a_j) = \sum_{i=1}^{S} U(s_i, a_j)\, p(s_i)$$

**Study task**

To see if you understand the rational approach under partial uncertainty and can calculate expected utility, attempt the following Self-test questions and then check your answers:

- Completion statement questions 4-6
- Multiple choice question 7

**A decision tree approach**

As an alternative to using a payoff matrix approach, the same problem can be structured and solved using a decision tree, Figure 3-1.

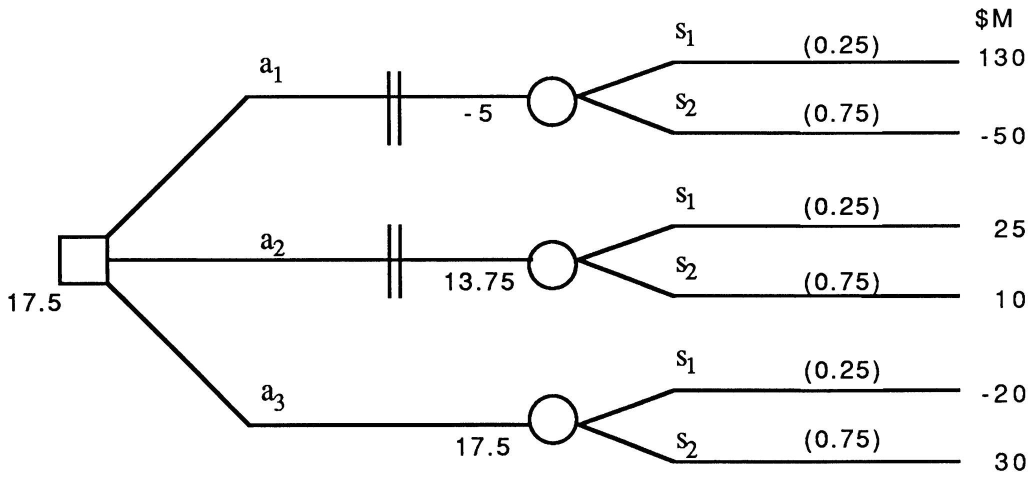



**Figure 3-1:** Decision tree for motor car manufacturer

Decision trees are very useful for analysing and structuring complex decision situations. A square is used to represent an action choice and a circle a chance event. Reading from left to right, the square is a decision point where one of three possible actions present themselves ($a_1$: large car; $a_2$: medium car; $a_3$: small car). If $a_1$ is selected, the circle indicates a chance event, the future prices of petrol ($s_1$: decrease; $s_2$: increase) with their probability estimates, and conditional payoffs of \$130M and -\$50M. Similarly, the other two sections represent the possibilities of selecting $a_2$ or $a_3$.

The decision is analysed by working backwards through the tree (folding back the tree). The expected utility of -\$5M at the chance event node on the $a_1$ path is calculated by multiplying the conditional payoffs by the probabilities of the states leading to them, and summing:

$$(130)(0.25) + (-50)(0.75) = -5,$$

exactly as was calculated from the payoff matrix. On the $a_2$ limb we get

$$(25)(0.25) + (10)(0.75) = 13.75,$$

and for the $a_3$ limb $(-20)(0.25) + (30)(0.75) = 17.5.$

These expected utilities are then ranked to find the highest. Paths leading to inferior payoffs are struck out. In this case, 17.5 is the maximum expected utility, corresponding to the $a_3$ path, so $a_3$ dominates $a_1$ and $a_2$. Therefore the $a_1$ and $a_2$ actions are struck out (double stroked). Finally, the 17.5 is transferred to the starting point. Thus the tree illustrates that $a_3$ is the optimum action, with an expected utility of $17.5M.

## Study task

Test your knowledge of decision trees by attempting the following Self-test questions and then check your answers:

- Matching questions 4-5
- Completion statement question 7

**Sensitivity analysis**

Based on the motor car manufacturer's probability estimates with respect to future petrol prices the optimum action has been identified as make a small car. However, what if his probability estimates are astray? It might be interesting to know whether a small change in the estimates would affect the choice of the optimum action, or whether substantial variations in probability do not alter the choice. In other words, how sensitive is the decision to the state probability estimates? Below is a condensed version of the payoff matrix, but without specific state probabilities; instead, p = the probability of a price increase, and consequently 1-p = the probability of a price decrease.

| | $p(s_1)=1-p$ | $p(s_2)=p$ |
|---|---|---|
| | $s_1$ | $s_2$ |
| $a_1$ | $130 | -$50 |
| $a_2$ | $25 | $10 |
| $a_3$ | -$20 | $30 |

| | | | |
|---|---|---|---|
| Now, $a_1$ dominates $a_2$ if | $E(U\|a_1)$ | $>$ | $E(U\|a_2)$, |
| | i.e. if $130(1-p) - 50p$ | $>$ | $25(1-p) + 10p$ |
| | $-165p$ | $>$ | $-105$ |
| | $p$ | $<$ | $0.636$ |
| Similarly, $a_1$ dominates $a_3$ if | $130(1-p) - 50p$ | $>$ | $-20(1-p) + 30p$, |
| | or if $p$ | $<$ | $0.652$ |
| Finally, $a_2$ dominates $a_3$ if | $25(1-p) + 10p$ | $>$ | $-20(1-p) + 30p$, |
| | or if $p$ | $<$ | $0.692$ |
| Summarising, $a_1$ dominates if | $p$ | $<$ | $0.636$ |
| $a_3$ dominates if | $p$ | $>$ | $0.692$ |
| $a_2$ dominates if | $0.636 < p$ | $<$ | $0.692$ |

Therefore our motor car manufacturer's estimate of $p(s_2) = 0.75$ can drop to 0.692 before there is any change in the optimum action, make a small car. When $p(s_2)$ lies between 0.636 and 0.692 a medium car is optimal, and once $p(s_2)$ falls below 0.636 a large car is the optimum choice.
Such supporting information gives the DM a feel for the impact of uncertainty in state probability estimates on action choice. Clearly, unless the chance of a price increase is substantially over 60%, a large car is his rational choice, and once it increases to 70% or more a small car is optimal.

## Utility functions

It was assumed in the illustrative problem that the DM had a **linear utility function**, indicating constant absolute risk aversion. Hence the utility of a gamble to the DM is independent of his wealth; he would prefer $a_3$ to $a_1$ or $a_2$ regardless of the size of his initial endowment. Most people are risk averse. They have **concave utility functions**, exhibiting decreasing absolute risk aversion - the higher one's initial wealth the less risk averse is one's behaviour. Or to put it another way, the higher one's initial wealth the less the utility gained from a given dollar return; this is the well known principle of **diminishing marginal utility**.

The **logarithmic utility function** is one example of a utility function reflecting risk averse attitudes. If our DM's preferences were characterised by a logarithmic function of the form

$$U(W) = \log_{10}(W)$$

it would be desirable to present the payoffs in terms of terminal wealth and take the logarithm of them.

For example, if his initial wealth were \$200M, his terminal wealth from selecting $a_1$ would be either \$330M ($s_1$) or \$150M ($s_2$). The payoff matrix would contain the logarithms of such terminal wealth positions, and would look like this:

| | $p(s_1)=0.25$ | $p(s_2)=0.75$ |
|---|---|---|
| | $s_1$ | $s_2$ |
| $a_1$ | log 330M = 8.5185 | log 150M = 8.1761 |
| $a_2$ | log 225M = 8.3522 | log 210M = 8.3222 |
| $a_3$ | log 180M = 8.2553 | log 230M = 8.3617 |

Consequently the expected utility calculations would result in:

$$E(U|a_1) = 8.5185(0.25) + 8.1761(0.75) = 8.2617$$
$$E(U|a_2) = 8.3522(0.25) + 8.3222(0.75) = 8.3297$$
$$E(U|a_3) = 8.2553(0.25) + 8.3617(0.75) = 8.3351 \ *$$

By experimenting with different initial endowments the reader will discover that in some circumstances $a_3$ may not be optimal. For example, given zero initial wealth, $a_2$ is optimal for the motor car manufacturer if his utility function is logarithmic as shown.

The important point to remember is that, unless the DM has a linear utility function the payoffs should be expressed in terms of utility. Further, it is often easier to work with terminal wealth (initial endowment plus incremental payoff) rather than with the incremental payoff, because the analysis can be performed in terms of total utility rather than having to calculate marginal utility.

From a practical viewpoint, how is a person's utility function established? It is possible to construct an approximation via standard reference lotteries. In these, the subject (client?) provides indifference judgements between a sure thing and a two-outcome lottery.

**Example**: 'Which would you prefer: \$30 000 for certain, or a lottery which returns either \$100 000 with probability (p) or \$0 with probability (1-p)?' If the certain sum is given (as above) the consultant keeps changing the probability until the subject registers indifference between the \$30 000 for certain and the lottery. Clearly, if the subject has a linear utility function she would register indifference when p is set at 0.3 [\$100 000(0.3) + \$0(0.7) = \$30 000]. If, however, the subject is risk averse she would require a higher probability than 0.3 before she would prefer the lottery to the certainty of \$30 000.

The probability at which the subject registers indifference represents her utility of the certain sum. A few repetitions in which the certain sum is varied enable a graph of the individual's utility function to be drawn. This graph can be used to read off the utility of given sums of money. An alternative approach is to present the subject with a given probability and vary the certain sum until the subject registers indifference.

### Study task

Complete the following Self-test questions concerning sensitivity analysis and utility functions and then check your answers:

- Matching questions 6-9
- Completion statement questions 8-10
- Multiple choice questions 2, 8

## Marginal analysis

Marginal analysis provides a quick method for determining the optimum action, especially when there is a large number of actions and states. In order to demonstrate marginal analysis, and compare it with expected utility calculations, the following example will be used.

Consider the problem of a shopkeeper who sells, among other things, meat pies. She buys the pies for $1.00 each and sells them for $1.80. Any pies left over at the end of the day are sold for pig food, for 40 cents each. The demand for pies is relatively constant over time, but varies from day to day. The shopkeeper wants to establish a standard daily order quantity which will maximise her long run profits from pies.

On examining past records she discovers that (for the sake of simplicity) her daily sales have been as shown in Table 3-4.

**Table 3-4:** Pie sales over last 100 days

| Daily Demand | Frequency | Relative Frequency (Probability) |
|---|---|---|
| 50 | 10 | 0.1 |
| 75 | 30 | 0.3 |
| 100 | 30 | 0.3 |
| 125 | 20 | 0.2 |
| 150 | 10 | 0.1 |
| | 100 | 1.0 |

## Solution using a payoff matrix

There are five possible states of nature (levels of demand), and accordingly there are five logical actions to consider: order 50, 75, 100, 125 or 150 pies daily. From this information a payoff matrix can be constructed, as in Table 3-5.

**Table 3-5:** Payoff matrix for pie example

| | 0.1 | 0.3 | 0.3 | 0.2 | 0.1 |
|---|---|---|---|---|---|
| **Demand:** | **$s_1$: 50** | **$s_2$: 75** | **$s_3$: 100** | **$s_4$: 125** | **$s_5$: 150** |
| **Order** | | | | | |
| **$a_1$: 50** | $40.00 | $40.00 | $40.00 | $40.00 | $40.00 |
| **$a_2$: 75** | 25.00 | 60.00 | 60.00 | 60.00 | 60.00 |
| **$a_3$: 100** | 10.00 | 45.00 | 80.00 | 80.00 | 80.00 |
| **$a_4$: 125** | (5.00) | 30.00 | 65.00 | 100.00 | 100.00 |
| **$a_5$: 150** | (20.00) | 15.00 | 50.00 | 85.00 | 120.00 |

In Table 3-5 the actions are shown in the rows (order 50,75 etc) while the states are in the columns (demand is 50, 75 etc).

- If the action *order 50 each day* is adopted the result will always be a profit of $40.00 (50 pies sold for 80c profit on each: $1.80 - $1.00 = $0.80). No matter how many are demanded she has no more than 50 pies to sell.
- Should she decide to order 75 each day, the possible results appear in the second row. Ordering 75 and selling 50 (column 1) results in $25.00 profit ($40.00 profit from selling 50, less $15.00 loss suffered on the unsold 25 - 60c loss on each). An alternative calculation is: sales revenue would be (50 x $1.80) + (25 x $0.40) = $90 + $10 = $100; costs would be 75 x $1 = $75; hence profit = $100 - $75 = $25. For a demand of 75 (column 2) or higher demands (columns 3 to 5) the profit will be $60.00 from selling all 75 pies.
- The third row shows the conditional results from ordering 100 pies daily. When the demand is 50, a profit of $10.00 results ($40.00 profit from selling 50 less $30.00 loss on the unsold 50). A demand of 75 produces a profit of $45.00 ($60.00 from sales less a loss of $15.00 on the unsold 25). Demands of 100, 125 or 150 give $80.00 profit on the sale of all 100 pies ordered.

Similarly the remainder of the table is completed. The profits on the main diagonal are those which would be gained with perfect knowledge.

The expected monetary profit of each strategy can be determined in the usual way. Thus the expected profit from ordering 100 pies is

$$\$10.00(0.1)+\$45.00(0.3)+\$80.00(0.3)+\$80.00(0.2)+\$80.00(0.1) = \$62.50.$$

The expected profits for each action are shown in Table 3-6.

**Table 3-6:** Expected profits

| Order | Expected Profit |
|---|---|
| 50 | $40.00 |
| 75 | 56.50 |
| 100 | 62.50 * |
| 125 | 58.00 |
| 150 | 46.50 |

The optimum action is to order 100 pies each day, resulting in a long run average daily profit of \$62.50.

### Solution using marginal analysis – discrete probability distribution

Let us now find the solution to the same problem using marginal analysis. Let p = the probability of selling an additional unit. Then (1-p) = the probability of not selling one additional unit. Let MP = the marginal profit earned by each unit and ML = the marginal loss suffered from not selling an additional unit.

For each additional unit ordered, the expected increase in profits is p(MP), the probability of selling the additional unit times its marginal profit, while the expected reduction in profits is (1-p)(ML), the probability of not selling it by the unit marginal loss.

As more and more units are added to the order size p decreases, and thus p(MP), the expected increase in profits, declines, whereas (1-p) increases, as does the expected reduction in profits, (1-p)(ML).

Additional units should be ordered provided p(MP) exceeds (1-p)(ML). The DM is indifferent between including or excluding an additional unit when

$$\begin{aligned} p(MP) &= (1-p)(ML) \\ &= ML - p(ML) \\ \text{i.e.} \quad p(MP) + p(ML) &= ML \\ \text{or} \quad p(MP + ML) &= ML \end{aligned}$$

$$\text{and hence} \quad p^* = \frac{ML}{MP + ML}$$

where p* indicates the required minimum probability that an additional unit will be sold to justify adding it to the order quantity. In fact, one should be **indifferent** if the probability of selling an additional unit is only equal to p*; it should be greater than p*.

In our example MP = \$1.80 - \$1.00 = 80c, and ML = \$1.00 - \$0.40 = 60c.

$$\text{Thus} \quad p^* = \frac{ML}{MP + ML} = \frac{60}{80 + 60} = \frac{60}{140} = 0.4285.$$

The rule is that additional units should be added to the order size if the probability of selling the order quantity or more is equal to or greater than p*. Therefore we need to determine the cumulative probability of selling various quantities **or more**, Table 3-7.

**Table 3-7:** Cumulative probability distribution for demand for pies

| Demand | Probability | Cumulative Probability |
|---|---|---|
| 50 | 0.1 | 1.0 |
| 75 | 0.3 | 0.9 |
| 100 | 0.3 | 0.6 |
| 125 | 0.2 | 0.3 |
| 150 | 0.1 | 0.1 |

In Table 3-7 we see that the probability of selling 50 or more pies is 1 (a certainty). The probability of selling 75 or more is 0.9. The probabilities fall as the order size is increased. Following the rule, find the cumulative probability (of selling the given number or more) which just equals or exceeds p*, or 0.4285 for this example[2]. The cumulative probability which just exceeds 0.4285 is 0.6, and this corresponds to a demand of 100 units. Therefore 100 units is the optimum order size. The shopkeeper would not increase the order size to 125 pies because the probability of selling 125 or more is only 0.3 (less than p*). She would increase the order size beyond 75 because the probability of selling 100 or more still exceeds p* and hence more profit results.

In summary, marginal analysis is a quicker method for finding the optimum action in problems of this nature. All that is required is

1. calculate p*;
2. convert the probability distribution into a cumulative one; and
3. find the cumulative probability which just equals or exceeds p*.

What marginal analysis doesn't provide is the expected profit from the optimum action. To find this, payoff matrix cell values for the optimum action are required and expected value calculations determine the expected profit.

## Solution using marginal analysis - continuous probability distributions

### Note

If you are not familiar with the normal distribution you should postpone reading this last section until you have studied the next topic on the normal distribution.

The previous pie example involved a discrete probability distribution for demand, assuming that sales could take only a few discrete values. In some cases it may be more realistic to make use of a continuous distribution.

2 Note that if the cumulative probability exactly equals p* then the DM would be indifferent between that order size and the next smaller one – they would have equal expected profit.

Using the same example where p* = 0.4285, assume that daily demand for pies is approximately normally distributed with a mean of 100 units and a standard deviation of 30 units. Figure 3-2 illustrates the situation. The shaded right tail shows the calculated value of p, 0.4285. The optimum order point is at X, so that the probability of selling X units or more is 0.4285.

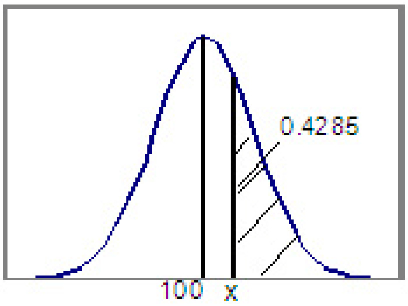



**Figure 3-2:** Demand for pies

The value of the point X can be found with the help of Table A (areas under the normal curve – found at the end of Topic 4). Table A actually gives the probability associated with the unshaded area of Figure 3-2. So we subtract 0.4285 from 1 to give 0.5715. Now we search the body of Table A to find the closest value to 0.5715, which is 0.57142. This value indicates a point which is 0.18 standard deviations to the right of the mean.

Alternatively, instead of using Table A the NORMSINV function of the Excel spreadsheet package can be used:

=NORMSINV(probability) returns the number of standard deviations from the mean associated with a cumulative probability less than or equal to the point x. Since we have the cumulative probability greater than or equal to x we need to subtract from 1:

> =NORMSINV(1-0.4285) returns the value 0.180194

This is slightly more accurate than the rounded value found in table A.

Thus the optimum order size is:

100 + 0.18 standard deviation
= 100 + 0.18(30)
= 105.4 pies

Since we cannot order 0.4 of a pie, the answer should be reduced to 105 pies (always round down because p* was the minimum probability).

### Study tasks

Complete the following Self-test questions and then check your answers:

- Matching question 10
- Multiple choice questions 9-10

## Additional Solved Problem

Yoggy Yoghurt Company is a small manufacturer of natural yoghurt that is sold to retailers. The weekly demand for the yoghurt varies from 3 cases to 6 cases. Each case costs $40 and sells for $100. Any cases unsold by the end of the week have no value due to spoilage.

*Required:*

1. How many cases should be manufactured each week if Yoggy follow the
   i. maximax criterion
   ii. maximin criterion
   iii. minimax regret criterion
   iv. Laplace criterion?

2. The probability that the demand will be 3 cases is 0.2, 4 cases 0.4, 5 cases 0.3 and 6 cases 0.1. Based on maximising EMV, how many cases should be manufactured each week?

3. Calculate the answer to (2) using marginal analysis.

**Solution**

First prepare a payoff matrix in dollars of profit per week:

| Make | D = 3 | D = 4 | D = 5 | D = 6 |
|---|---|---|---|---|
| 3 | 180 | 180 | 180 | 180 |
| 4 | 140 | 240 | 240 | 240 |
| 5 | 100 | 200 | 300 | 300 |
| 6 | 60 | 160 | 260 | 360 |

If demand is 3 cases and 3 are made profit is $180 (3 x $60) [SP $100 – CP $40 = $60]. This will be the profit from making 3 cases no matter what the demand.

If 4 cases are made and only 3 demanded profit will be $180 - $40 = $140 [$180 from the 3 that are sold less $40 loss on the unsold 4th case]. If demand is 4 or more the profit is $240 [4 x $60]. similarly the other cells are filled in.

1.a. The maximax criterion selects the row with the highest or maximum profit, row 4, make 6 cases:

| Make | Max | Maximax |
|---|---|---|
| **3** | 180 | |
| **4** | 240 | |
| **5** | 300 | |
| **6** | 360 | Best |

1.b. The maximin criterion establishes the minimum profit for each strategy and then selects the maximum of these worst possible results, make 3 cases:

| Min | Maximin |
|---|---|
| 180 | Best |
| 140 | |
| 100 | |
| 60 | |

1.c. First a regret matrix is formed. Then the maximum regret for each action is established. Finally, the minimum of these maximum regrets is chosen, make 5 cases:

**Regret Matrix**

| Make | D = 3 | D = 4 | D = 5 | D = 6 | Max | Minimax |
|---|---|---|---|---|---|---|
| **3** | 0 | 60 | 120 | 180 | 180 | |
| **4** | 40 | 0 | 60 | 120 | 120 | |
| **5** | 80 | 40 | 0 | 60 | 80 | Best |
| **6** | 120 | 80 | 40 | 0 | 120 | |

1.d. The Laplace criterion assumes each state is equally likely. The solution involves finding the average EMV for each action and selecting the maximum, make 5 cases.

| Average | LaPlace |
|---|---|
| 180 | |
| 215 | |
| 225 | Best |
| 210 | |

2. The EMV for each action is calculated by multiplying the profits in each row by the probabilities and summing. The action with the maximum EMV is selected. In this case there is a tie between making 4 or 5 cases, meaning that either would be the optimum choice, both giving an expected profit of $220:

| EMV | Choice |
|---|---|
| 180 | |
| 220 | Best |
| 220 | Best |
| 190 | |

3. MP = \$100 - \$40 = \$60
ML = \$40

$$p^* = \frac{ML}{MP + ML} = \frac{40}{60 + 40} = 0.4$$

| Demand | Probability | Cumulative Probability |
|---|---|---|
| 3 | 0.2 | 1.0 |
| 4 | 0.4 | 0.8 |
| 5 | 0.3 | 0.4 |
| 6 | 0.1 | 0.1 |

Now we look to see what cumulative probability equals or exceed p* of 0.4. In this case the cumulative probability of 0.4 corresponds with 5 cases. Since the cumulative probability is exactly equal to p* the next smallest quantity is also optimal, i.e. 4 cases. So we conclude that the optimum number to make is either 4 or 5 cases, exactly as determined by the EMV calculations in 2.

## Using a Spreadsheet for Decision Analysis

Although you need to be able to perform the calculations manually you might like to build a spreadsheet model to check your answers. Below is an Excel spreadsheet model output for the Motor Car Manufacturer illustrative example in this topic. Compare the output with the calculations in the text.

| | A | B | C | D | E | F | G | H | I |
|---|---|---|---|---|---|---|---|---|---|
| 1 | | States of Nature | | | | | | | |
| 2 | Probability: | 0.25 | 0.75 | | | | | | |
| 3 | Make | Price Inc | Price Dec | | | | | | |
| 4 | Large car | 130 | -50 | | | | | | |
| 5 | Medium car | 25 | 10 | | | | | | |
| 6 | Small car | -20 | 30 | | | | | | |
| 7 | | | | | | | | | |
| 8 | | | | | | | | | |
| 9 | | | | | | | | | |
| 10 | Make | Max | Maximax | Min | Maximin | Average | LaPlace | EMV | Choice |
| 11 | Large car | 130 | Best | -50 | | 40 | Best | -5 | |
| 12 | Medium car | 25 | | 10 | Best | 17.5 | | 13.75 | |
| 13 | Small car | 30 | | -20 | | 5 | | 17.5 | Best |
| 14 | | | | | | | | | |
| 15 | | | | | | | | | |
| 16 | | | | | | | | | |
| 17 | | | REGRET MATRIX | | | | | | |
| 18 | | States of Nature | | | | | | | |
| 19 | Probability: | 0.25 | 0.75 | | | | | | |
| 20 | Make | Price Inc | Price Dec | Max | Minimax | | | | |
| 21 | Large car | 0 | 80 | 80 | Best | | | | |
| 22 | Medium car | 105 | 20 | 105 | | | | | |
| 23 | Small car | 150 | 0 | 150 | | | | | |

The formulas appear below. Rows 10 to 13 and 15 to 21 have been split up to fit on the page:

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 1 | | States of Nature | | | |
| 2 | Probability: | 0.25 | 0.75 | | |
| 3 | Make | Price Inc | Price Dec | | |
| 4 | Large car | 130 | -50 | | |
| 5 | Medium car | 25 | 10 | | |
| 6 | Small car | -20 | 30 | | |
| 7 | | | | | |
| 8 | | | | | |
| 9 | | | | | |
| 10 | Make | Max | Maximax | Min | Maximin |
| 11 | Large car | =MAX(B4:C4) | =IF(B11=MAX($B$11:$B$13),"Best","") | =MIN(B4:C4) | =IF(D11=MAX($D$11:$D$13),"Best","") |
| 12 | Medium car | =MAX(B5:C5) | =IF(B12=MAX($B$11:$B$13),"Best","") | =MIN(B5:C5) | =IF(D12=MAX($D$11:$D$13),"Best","") |
| 13 | Small car | =MAX(B6:C6) | =IF(B13=MAX($B$11:$B$13),"Best","") | =MIN(B6:C6) | =IF(D13=MAX($D$11:$D$13),"Best","") |

| | F | G | H | I |
|---|---|---|---|---|
| 10 | Average | LaPlace | EMV | Choice |
| 11 | =AVERAGE(B4:C4) | =IF(F11=MAX($F$11:$F$13),"Best","") | =SUMPRODUCT(B4:F4,$B$2:$F$2) | =IF(H11=MAX($H$11:$H$13),"Best","") |
| 12 | =AVERAGE(B5:C5) | =IF(F12=MAX($F$11:$F$13),"Best","") | =SUMPRODUCT(B5:F5,$B$2:$F$2) | =IF(H12=MAX($H$11:$H$13),"Best","") |
| 13 | =AVERAGE(B6:C6) | =IF(F13=MAX($F$11:$F$13),"Best","") | =SUMPRODUCT(B6:F6,$B$2:$F$2) | =IF(H13=MAX($H$11:$H$13),"Best","") |

| | A | B | C | D | E |
|---|---|---|---|---|---|
| 15 | | | REGRET MATRIX | | |
| 16 | | States of Nature | | | |
| 17 | Probability: | =B2 | =C2 | | |
| 18 | Make | Price Inc | Price Dec | Max | Minimax |
| 19 | Large car | =MAX(B$4:B$6)-B4 | =MAX(C$4:C$6)-C4 | =MAX(B19:C19) | =IF(D19=MIN($D$19:$D$21),"Best","") |
| 20 | Medium car | =MAX(B$4:B$6)-B5 | =MAX(C$4:C$6)-C5 | =MAX(B20:C20) | =IF(D20=MIN($D$19:$D$21),"Best","") |
| 21 | Small car | =MAX(B$4:B$6)-B6 | =MAX(C$4:C$6)-C6 | =MAX(B21:C21) | =IF(D21=MIN($D$19:$D$21),"Best","") |

Now build this model yourself. Cell ranges can be changed to accommodate smaller or larger number of states of nature and or actions.

**Read**

Reading 3.1: Tversky, A., & Kahneman, D. (1977). Judgement under uncertainty: Heuristics and biases. In G. M. Kahneman & H. Thomas (Eds.), *Modern decision analyzing* (pp. 39-61). Penguin.

This article is a much quoted article in the literature on choice. The authors claim that when people make decisions they frequently use simplifying techniques, or **heuristics** (rules of thumb)**.** Tversky and Kahneman suggest and illustrate **three** heuristics employed by people. You should understand and learn what each of these three heuristics are, and be aware of the biases or faulty estimates that can result from using such heuristics.

**From theory to practice**

Have you ever been fallen for the traps of errors in judgment caused by reliance on the heuristics of **representativeness** or **availability** described by Tversky and Kahneman?

## Summary

This topic was concerned with decision making under various conditions of uncertainty. It first introduced the payoff matrix for a specific illustrative example which formed the basis of introducing various criteria for selecting one of the proposed actions for implementation.

Under conditions of complete uncertainty (no knowledge of the probability of occurrence of the states) the DM can follow various criteria for selecting an action. The particular criterion adopted by a DM depends upon personal preferences and attitudes to risk. Four criteria were explored: maximax criterion, maximin criterion, minimax regret criterion and Laplace criterion.

Under conditions of partial uncertainty the DM can estimate state probabilities and follow the rational model of choice by maximising expected utility. It was assumed that the DM is risk neutral with a linear utility function. Marginal analysis was also introduced as a quick way to determine the optimum action when there is a large number of actions and states.

**Review task**

Check your understanding of decisions under complete and partial uncertainty by attempting Question Bank 3-20 and then check your answer.

## Self-test questions

### I Matching

You are required to match the numbered term with the letter of the most appropriate description.

| | | | | |
|---|---|---|---|---|
| 1. | _____ | Pessimism | A | When state probabilities are available we refer to the decision making process as decision making under ____________ . |
| 2. | _____ | Optimism | B | Under complete uncertainty the maximax criterion, when payoffs are profits, reflects this trait. |
| 3. | _____ | Laplace | C | Under complete uncertainty the maximin criterion, when payoffs are profits, reflects this trait. |
| 4. | ____ | Partial uncertainty | D | When it is assumed that states are equally likely a decision maker would use this criterion. |
| 5. | _____ | Decision tree | E | This tool is an alternative to a payoff matrix when analysing decisions. |
| 6. | _____ | Risk averse | F | A method for testing whether an optimum action is likely to change with small variations in state probability estimates. |
| 7. | _____ | Risk seeking | G | The utility from a gain of $100 is equal to the disutility from a loss of $100 for this decision maker. |
| 8. | _____ | Risk neutral | H | The utility from a gain of $100 is less than the disutility from a loss of $100 for this decision maker. |
| 9. | _____ | Sensitivity analysis | I | A technique for solving some problems with a large number of alternatives and states. |
| 10. | _____ | Marginal analysis | J | The utility from a gain of $100 is greater than the disutility from a loss of $100 for this decision maker. |

## II Completion statements

Complete each of the following statements with the most appropriate word or words.

1. Certainty exists when there is absolutely no doubt about ________________.
2. Because available information for a decision relates to the future it ________.
3. Uncertainty in decision making is usually assumed to relate only to _______.
4. Objective probabilities are based on _________________________________.
5. Subjective probabilities are _________________________________________.
6. The rational DM's rule is to _______________________________________.
7. _________________________________ is an expression used to describe the calculation of expected values and the determination of the best action from a *decision tree*.
8. A 'what if' technique that measures how the predictions of a decision model will be affected by changes in the critical data inputs is called __________________.
9. The logarithmic utility function reflects a _____________________________ attitude.
10. A good decision does not guarantee _________________________________.

## III Multiple choice

For each of the following questions identify the correct alternative:

1. Under complete uncertainty, given a decision matrix where the consequences are expressed in dollar costs, a pessimist would follow:

   A the maximax criterion
   B the maximin criterion
   C the minimax criterion
   D the minimin criterion
   E none of the above

2. Faced with the same decision situation, two rational decision makers would make the same decision if they were both:

   A risk averse
   B risk neutral
   C risk seeking
   D equally wealthy
   E none of the above

**Use the following conditional profits matrix to answer Questions 3 to 7:**

| | States | | | | |
|---|---|---|---|---|---|
| **Actions** | **1** | **2** | **3** | **4** | **5** |
| **a** | -3 | 5 | 7 | 4 | 8 |
| **b** | -4 | 6 | 3 | 1 | 5 |
| **c** | -2 | 4 | 7 | 1 | 6 |
| **d** | -1 | 5 | 9 | 3 | 4 |
| **e** | 0 | 6 | 2 | 5 | 7 |

3. An optimist would select action:

   A a
   B b
   C c
   D d
   E e

4. A pessimist would select action:

   A a
   B b
   C c
   D d
   E e

5. The Laplace criterion would lead to selection of action:

   A a
   B b
   C c
   D d
   E e

6. The minimax regret criterion indicates action:

   A a
   B b
   C c
   D d
   E e

7. Suppose the state probabilities are:

| State | Probability |
|---|---|
| 1 | 0.2 |
| 2 | 0.1 |
| 3 | 0.3 |
| 4 | 0.3 |
| 5 | 0.1 |
| | 1.0 |

Based on expected monetary values, the optimum action is:

A a
B b
C c
D d
E e

8. You are considering an expansion of your small business which currently earns an annual net income of $75,000. If expansion is successful net income should increase to $150,000, but if unsuccessful net income would contract to $30,000. [All figures are after tax.]

If you are risk neutral, the critical probability at which you would be indifferent between expanding or not expanding is:

A p(success) = 0.625
B p(success) = 0.500
C p(success) = 0.375
D p(success) = 0.200
E none of the above

9. Daily sales for a perishable food product are known to be 8,9,10 or 11 cases with probabilities of 0.2, 0.3, 0.4 and 0.1 respectively. Cases not sold during the day are worthless, but cases can only be produced in the morning before the store opens. The cost of producing one of these is $4 while the selling price is $7. How many cases should you produce to maximise profit?

A 8
B 9
C 10
D 11
E none of the above

10. Pies are ordered in advance by a vendor who sells them at football matches. Each pie costs 50 cents and is sold for $1.50. All unsold pies have to be thrown away at the end of the match. Demand is normally distributed, the expected number sold at each match being 1000 with a standard deviation of 148. If the pie vendor wishes to maximise his expected monetary payoff he will order for each match approximately

    A 1148 pies
    B 1063 pies
    C 1000 pies
    D 937 pies
    E 852 pies

## Self-test solutions

### I Matching

1. C  2. B  3. D  4. A  5. E  6. H  7. J  8. G  9. F  10. I

### II Completion statements

1. which state will occur
2. can never be perfect
3. state probabilities
4. objective, verifiable evidence
5. estimated by the DM
6. maximise expected utility
7. folding back the tree
8. sensitivity analysis
9. risk averse
10. a good outcome

### III Multiple choice

1. C
2. B
3. D Maximax: Action d has the maximum maximum of 9
4. E Maximin: Actions a to d all have a minimum of a negative amount.
5. A E(U|a) = (-3+5+7+4+8)/5 = 21/5 = 4.2 Every other action totals < 21.

6. A

Regret matrix

| | **1** | **2** | **3** | **4** | **5** | **Maximum** |
|---|---|---|---|---|---|---|
| **a** | 3 | 1 | 2 | 1 | 0 | 3 * Minimax regret |
| **b** | 4 | 0 | 6 | 4 | 3 | 6 |
| **c** | 2 | 2 | 2 | 4 | 2 | 4 |
| **d** | 1 | 1 | 0 | 2 | 4 | 4 |
| **e** | 0 | 0 | 7 | 0 | 1 | 7 |

7. D E(U|d)= -1(0.2)+5(0.1)+9(0.3)+3(0.3)+4(0.1) = 4.3 [maximum of all actions]

8. C

| | **Success** | **Failure** |
|---|---|---|
| Expand | $150,000 | $30,000 |
| Don't expand | 75,000 | 75,000 |

Let p= prob of Success and (1-p) = prob of Failure

Indifferent when E(Expand) = E(Don't expand)

i.e. when 150,000(p) + 30,000(1-p) = 75,000

120,000 p = 45,000

p = 45,000/120,000 = 0.375

9. B

| **Demand** | **Probability** | **Cumulative Probability** |
|---|---|---|
| 8 | 0.2 | 1.0 |
| 9 | 0.3 | 0.8 |
| 10 | 0.4 | 0.5 |
| 11 | 0.1 | 0.1 |

MP = $7 - $4 = $3 ML = $4
P* = ML/(MP+ML) = 4/(3+4) = 4/7 = 0.57
Cum prob just > 0.57 = 0.8, ie produce 9

10. B MP = $1.50 - $0.50 = $1.00 ML = $0.50
p* = ML/(MP+ML) = 50/(100+50) = 50/150 = 0.33333
Look up 0.666667 in Table A gives Z = 0.43
i.e. order 1000 + 0.43(148) = 1000 + 63.64 = 1063.64 = 1063 (round down)

## Review task

### Question Bank 3-20

a. Don't launch:

| | **Markem Price $6** | **Markem Price $7** |
|---|---|---|
| **$a_1$: Don't launch** | -$1000 | -$1000 |
| **$a_2$: Launch @ $7** | (7-3)(10 000)(0.01) - 8000 = -7600 | (7-3)(10 000)(0.25) - 8000 = 2000 |
| **$a_3$: Launch @ $6** | (6-3)(10 000)(0.05) - 8000 = -6500 | (6-3)(10 000)(0.9) - 8000 = 19000 |

$E(U|a_1) = -1000^*$
$E(U|a_2) = -7600(0.8) + 2000(0.2) = -5680$
$E(U|a_3) = -6500(0.8) + 19000(0.2) = -1400$

b. Maximax - Launch @ $6

c. Maximin - Don't Launch

d. Laplace: Launch @ $6:

$E(U|a_1) = -1000$
$E(U|a_2) = (-7600+2000)/2 = -2800$
$E(U|a_3) = (-6500+19000)/2 = 6250^*$

e. Minimax regret: a3: Launch @ $6:

| | **Regret Matrix** | | |
|---|---|---|---|
| | **Markem price: $6** | **Markem price: $7** | **Maximum** |
| **$a_1$** | 0 | 20 000 | 20 000 |
| **$a_2$** | 6600 | 17 000 | 17 000 |
| **$a_3$** | 5500 | 0 | 5 500 * |

## References

Hurcwicz, L. (1951). Optimality criteria for decision making under ignorance. Cowles Commission Discussion Paper, *Statistics No. 370*.

Savage, L. J. (1951). The theory of statistical decision. *Journal of the American Statistical Association*, *46*, 55-67.

Wald, A. (1945). Statistical decision functions which minimize the maximum risk. *Annals of Mathematics*, *46*, 265-280.

# Topic 4 Normal probability distribution

## Essential reading

**Textbook** Chapter 2, pp. 61-68.

## Objectives

At the end of this topic, you should be able to:

- describe the features of the normal distribution;
- calculate Z-values;
- calculate areas under the normal curve using Z-values and a normal table;
- apply these skills to simple decision problems.

## Commentary

In Topic 2 we discussed probability distributions, noting that they could be discrete or continuous. One of the most common and useful continuous probability distributions is the **normal** distribution. This topic is concerned solely with the use of the normal distribution and a table of areas under the normal curve. Some of you may be familiar with this area of statistics and may simply use this material as a refresher course, or you may skip it if you are confident of your ability in this area.

### Features of normal distribution

Many phenomena seem to follow a pattern of variation consistent with the normal distribution. The normal distribution is a family of distributions of the same basic mathematical form. They are all bell-shaped, and symmetrical, and have a single peak in the centre, as shown in Figure 4-1.

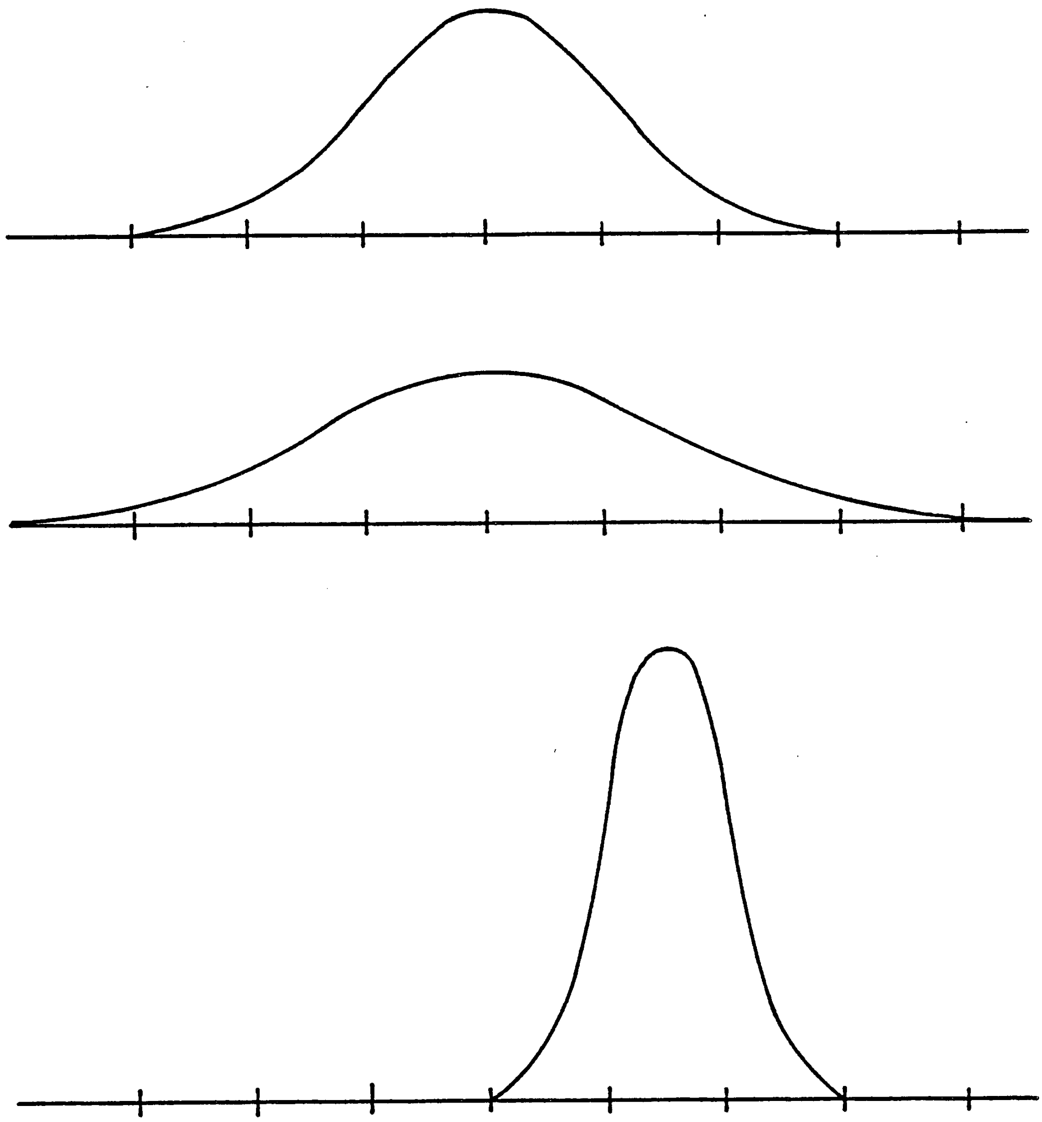


**Figure 4-1** Three different normal distributions

They vary, however, in the location of the average (or mean, or centre of the distribution) and in the extent of variability (the standard deviation). A normal distribution is completely specified when the value for the two parameters, the **mean** and the **standard deviation,** are known.

The area under the curve is a measure of probability, and the total area under the curve sums to 1. The spread of a normal distribution is such that:

- the area between the mean ±1 standard deviation is about 0.68;
- the area between the mean ±2 standard deviations is about .954;
- and the area between the mean ±3 standard deviations is about 0.997.

Thus an area bounded by 3 standard deviations either side of the mean represents 99.7% of the total area under the curve, indicating that most values in a normal distribution fall within 3 standard deviations of the mean. The above information is shown in Figure 4-2.

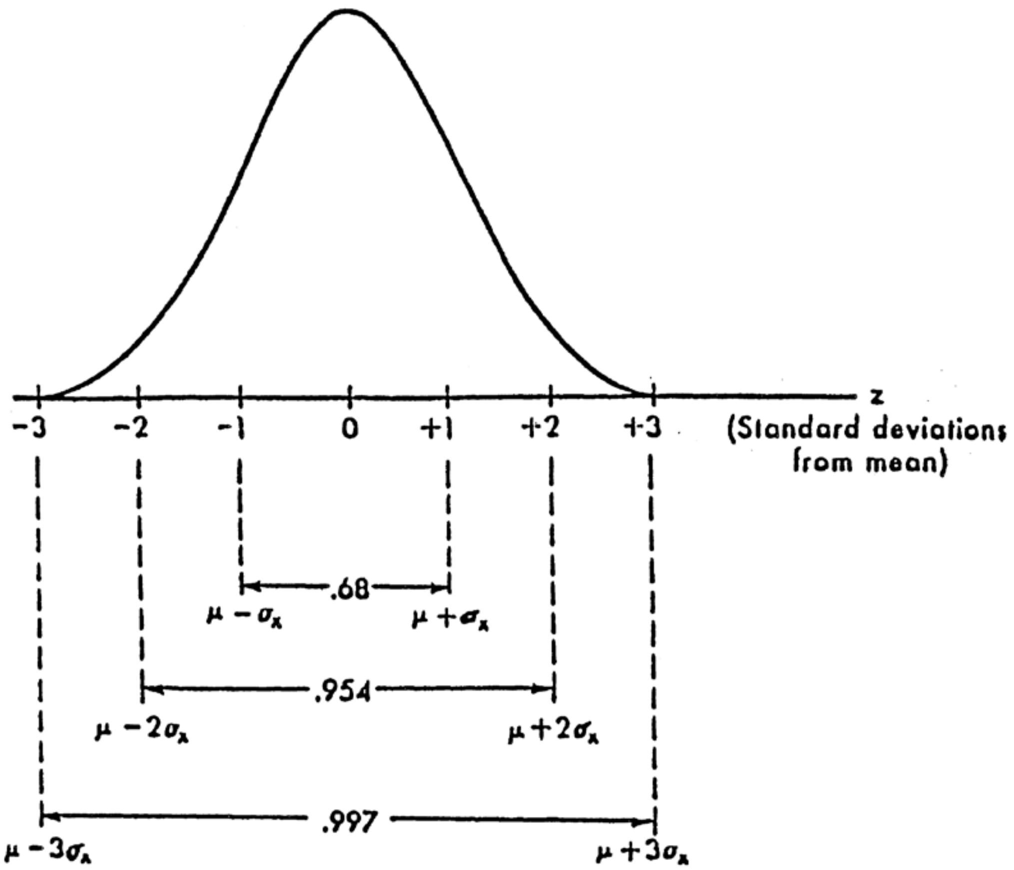



**Figure 4-2**

Another fact is that the area between the mean ± 2/3 of a standard deviation is 0.5. That is, 50% of the distribution lies within 2/3 of a standard deviation either side of the mean.

## Calculating normal probabilities using Z-values

Rather than have to use integral calculus to calculate area, and hence probabilities, we can use the **standard normal** table for probability estimates. A copy of this table, Table 4-1, appears on page 89. In order to use this table we must convert any normal distribution to a standard normal distribution - one which has a mean of zero and a standard deviation of one. This conversion uses the formula:

$$Z = \frac{X - \mu}{\sigma}$$

which simply calculates the Z-value, that is, the number of standard deviations a point X is from the mean $\mu$. Given the Z-value, we can refer to the standard normal table.

**Example.** To illustrate, suppose that the life of a light globe is normally distributed with a mean of 1,000 hours and a standard deviation of 200 hours. What is the probability that a globe chosen at random will have a life of 1,250 hours or less? To answer this question we need to find the shaded area under the curve in Figure 4-3.

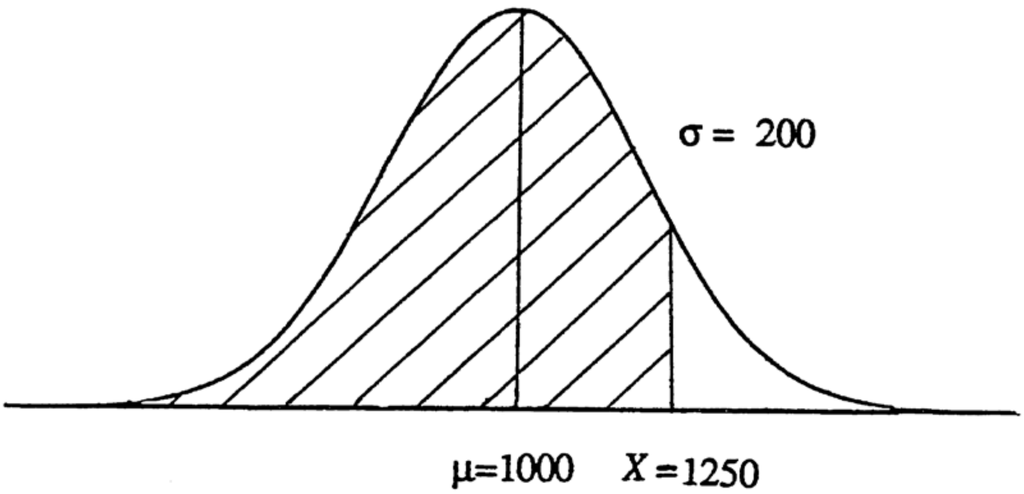



**Figure 4-3**

First we compute Z:

$$Z = \frac{X - \mu}{\sigma} = \frac{1{,}250 - 1{,}000}{200} = \frac{250}{200} = 1.25$$

Looking at Table A for a Z-value of 1.25, we find an area under the curve of 0.89435. We do this by looking up 1.2 in the left hand column of the table, and then moving across the 1.2 row to the .05 column. Thus, the probability of a globe having a life of 1,250 hours or less is 0.89435, about an 89% chance.

The same answer can be obtained from the Excel spreadsheet using the NORMDIST function:

=NORMDIST(x,mean,standard-dev,cumulative)

| | | |
|---|---|---|
| where x | = | sample value |
| mean | = | mean of the distribution |
| standard-dev | = | standard deviation of the normal distribution |
| cumulative | = | a logical value - TRUE is required here |

Thus, =NORMDIST(1250, 1000, 200, TRUE) returns the value 0.89435016 as determined from Table A in the manual process above. This value is the cumulative distribution from the left hand end up to the value of x. Hence the NORMDIST function returns the probability of a *sample value or less* from a normal distribution.

What is the probability of a globe lasting longer than 1,400 hours? Figure 4-4 illustrates this probability, in the shaded area.

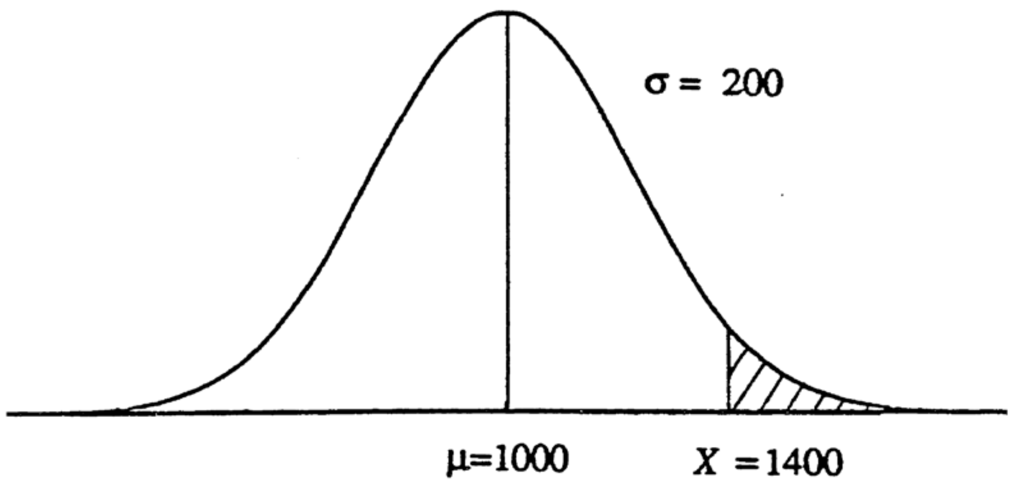



**Figure 4-4**

Now compute the Z-value:

$$\begin{aligned} Z &= \frac{X-\mu}{\sigma} \\ &= \frac{1{,}400-1{,}000}{200} \\ &= \frac{400}{200} \\ &= 2.0 \end{aligned}$$

If we look up 2.0 in Table A we get the value 0.97725. But this is not the shaded area of Figure 4-4, but rather it is the unshaded area. However, because the total area under the curve is equal to 1, the area of the shaded part is equal to (1 minus the unshaded area). In general, the probability that Z is greater than some value is equivalent to 1 minus the probability that Z is less than that value. Accordingly,

$$\begin{aligned} p(X > 1{,}400) &= p(Z > 2.0) \\ &= 1 - p(Z \le 2.0) \\ &= 1 - 0.97725 \\ &= 0.02275 \end{aligned}$$

Using Excel, we would have to calculate 1-NORMDIST to get the cumulative probability of a *sample value or more.*

```
=1-NORMDIST(1400,1000,200,TRUE)=0.02275
```

The probability that a globe's life will be less than 750 hours is represented by the shaded area in Figure 4-5.

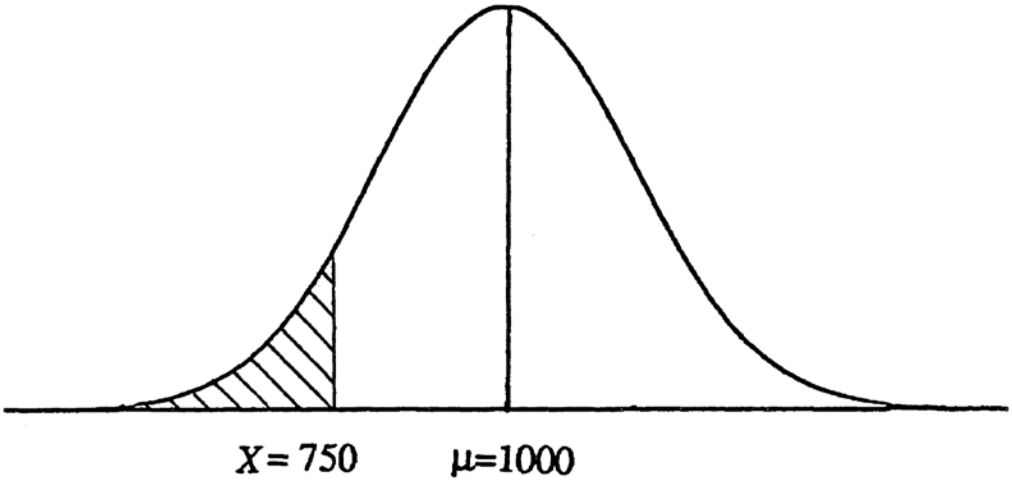



**Figure 4-5**

$$p(X < 750) = p\left[Z < \frac{750 - 1,000}{200}\right]$$
$$= p(Z < -1.25)$$

This Z-value is negative, but the standard normal table only has positive values of Z. To overcome this problem we observe that, because the curve is symmetrical, the probability that Z < -1.25 is equivalent to 1 minus the probability that Z < 1.25:

$$\begin{aligned} p(Z < -1.25) &= 1 - p(Z < 1.25) \\ &= 1 - 0.89435 \\ &= 0.10565 \end{aligned}$$

Using Excel, p(X < 750):

```
=NORMDIST(750,1000,200,TRUE) returns 0.10564984
```

The probability that a globe will last more than 900 hours is illustrated in Figure 4-6.

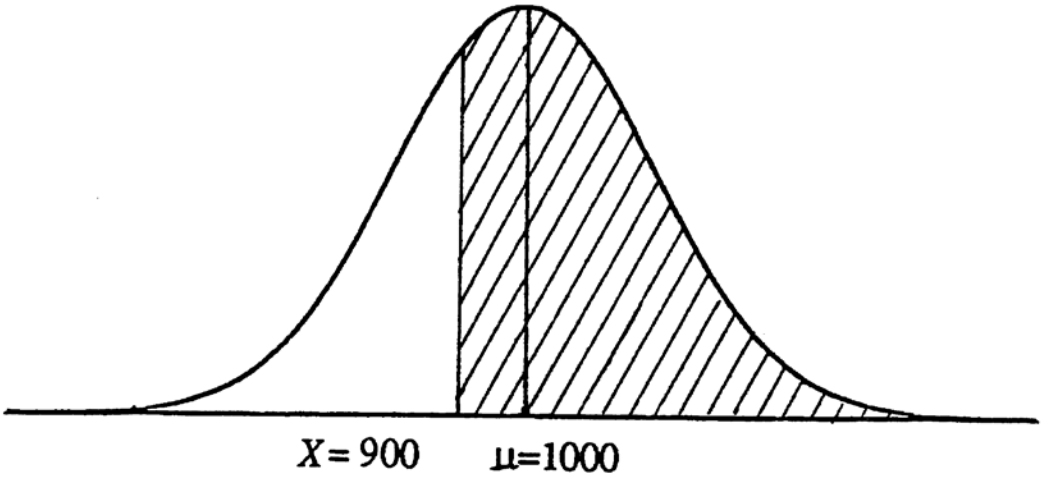



**Figure 4-6**

$$\begin{aligned} p(X > 900) &= p\left[Z > \frac{900 - 1{,}000}{200}\right] \\ &= p(Z > -0.5) \\ &= 1 - p(Z > 0.5) \\ &= p(Z < 0.5) \\ &= 0.69146 \end{aligned}$$

Using Excel, p(X > 900):

```
=1-NORMDIST(900,1000,200,TRUE) gives 0.6914625.
```

Finally, what is the probability that a globe's life will be between 1,100 and 1,200 hours?

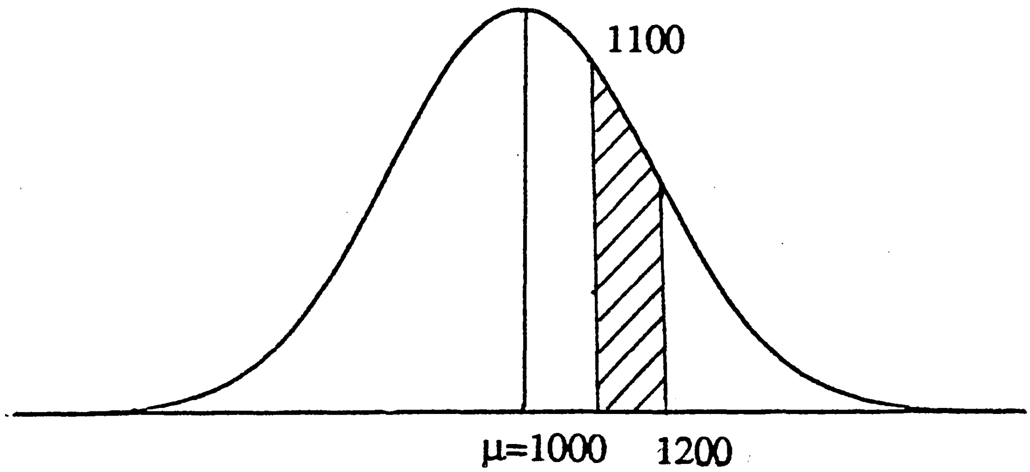



**Figure 4-7**

We see that:

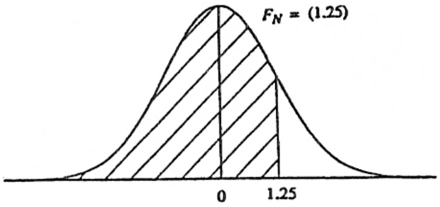
$$
\begin{aligned}
p(1{,}100 \le X \le 1{,}200) &= p(X \le 1{,}200) - p(X \le 1{,}100) \\
&= p[Z \le (1{,}200 - 1{,}000)/200)] - p(Z \le (1{,}100 - 1{,}000)/200] \\
&= p(Z \le 1.0) - p(Z \le 0.5) \\
&= 0.84134 - 0.69146 \\
&= 0.14988
\end{aligned}
$$

With Excel:

```
=NORMDIST(1200,1000,200,TRUE)-NORMDIST(1100,1000,200,TRUE)
returns the value 0.1498823.
```

Remember, because the normal distribution is symmetrical, and the total area under the curve sums to 1, each half of the distribution (left of the mean, or right of the mean) has an area equal to 0.5. It is wise to draw a diagram each time, and you can observe whether the shaded area is more or less than one half of the distribution. The advantage of this when using the manual procedure and Table A is that you know immediately whether you have correctly looked up the probability, or whether you need to subtract the table value from 1. If you have shaded less than half the area, yet the probability from the table is more than 0.5, you would realise that you must subtract from 1.

**Table A:** Areas under the normal curve




$$z = \frac{x-\mu}{\sigma}$$

Example: $p\,(z \le 1.25) = F_N(1.25)$
$= 0.89435$

| Z | .00 | .01 | .02 | .03 | .04 | .05 | .06 | .07 | .08 | .09 |
|---|---|---|---|---|---|---|---|---|---|---|
| 0.0 | .50000 | .50399 | .50798 | .51197 | .51595 | .51994 | .52392 | .52790 | .53188 | .53586 |
| 0.1 | .53983 | .54380 | .54776 | .55172 | .55567 | .55962 | .56356 | .56749 | .57142 | .57535 |
| 0.2 | .57926 | .58317 | .58706 | .59095 | .59483 | .59871 | .60257 | .60642 | .61026 | .61409 |
| 0.3 | .61791 | .62172 | .62552 | .62930 | .63307 | .63683 | .64058 | .64431 | .64803 | .65173 |
| 0.4 | .65542 | .65910 | .66276 | .66640 | .67003 | .67364 | .67724 | .68082 | .68439 | .68793 |
| 0.5 | .69146 | .69497 | .69847 | .70194 | .70540 | .70884 | .71226 | .71566 | .71904 | .72240 |
| 0.6 | .72575 | .72907 | .73237 | .73536 | .73891 | .74215 | .74537 | .74857 | .75175 | .75490 |
| 0.7 | .75804 | .76115 | .76424 | .76730 | .77035 | .77337 | .77637 | .77935 | .78230 | .78524 |
| 0.8 | .78814 | .79103 | .79389 | .79673 | .79955 | .80234 | .80511 | .80785 | .81057 | .81327 |
| 0.9 | .81594 | .81859 | .82121 | .82381 | .82639 | .82894 | .83147 | .83398 | .83646 | .83891 |
| 1.0 | .84134 | .84375 | .84614 | .84849 | .85083 | .85314 | .85543 | .85769 | .85993 | .86214 |
| 1.1 | .86433 | .86650 | .86864 | .87076 | .87286 | .87493 | .87698 | .87900 | .88100 | .88298 |
| 1.2 | .88493 | .88686 | .88877 | .89065 | .89251 | .89435 | .89617 | .89796 | .89973 | .90147 |
| 1.3 | .90320 | .90490 | .90658 | .90824 | .90988 | .91149 | .91309 | .91466 | .91621 | .91774 |
| 1.4 | .91924 | .92073 | .92220 | .92364 | .92507 | .92647 | .92785 | .92922 | .93056 | .93189 |
| 1.5 | .93319 | .93448 | .93574 | .93699 | .93822 | .93943 | .94062 | .94179 | .94295 | .94408 |
| 1.6 | .94520 | .94630 | .94738 | .94845 | .94950 | .95053 | .95154 | .95254 | .95352 | .95449 |
| 1.7 | .95543 | .95637 | .95728 | .95818 | .95907 | .95994 | .96080 | .96164 | .96246 | .96327 |
| 1.8 | .96407 | .96485 | .96562 | .96638 | .96712 | .96784 | .96856 | .96926 | .96995 | .97062 |
| 1.9 | .97128 | .97193 | .97257 | .97320 | .97381 | .97441 | .97500 | .97558 | .97615 | .97670 |
| 2.0 | .97725 | .97784 | .97831 | .97882 | .97932 | .97982 | .98030 | .98077 | .98124 | .98169 |
| 2.1 | .98214 | .98257 | .98300 | .98341 | .98382 | .98422 | .98461 | .98500 | .98537 | .98574 |
| 2.2 | .98610 | .98645 | .98679 | .98713 | .98745 | .98778 | .98809 | .98840 | .98870 | .98899 |
| 2.3 | .98928 | .98956 | .98983 | .99010 | .99036 | .99061 | .99086 | .99111 | .99134 | .99158 |
| 2.4 | .99180 | .99202 | .99224 | .99245 | .99266 | .99286 | .99305 | .99324 | .99343 | .99361 |
| 2.5 | .99379 | .99396 | .99413 | .99430 | .99446 | .99461 | .99477 | .99492 | .99506 | .99520 |
| 2.6 | .99534 | .99547 | .99560 | .99573 | .99585 | .99598 | .99609 | .99621 | .99632 | .99643 |
| 2.7 | .99653 | .99664 | .99674 | .99683 | .99693 | .99702 | .99711 | .99720 | .99728 | .99736 |
| 2.8 | .99744 | .99752 | .99760 | .99767 | .99774 | .99781 | .99788 | .99795 | .99801 | .99807 |
| 2.9 | .99813 | .99819 | .99825 | .99831 | .99836 | .99841 | .99846 | .99851 | .99856 | .99861 |
| 3.0 | .99865 | .99869 | .99874 | .99878 | .99882 | .99886 | .99889 | .99893 | .99896 | .99900 |
| 3.1 | .99903 | .99906 | .99910 | .99913 | .99916 | .99918 | .99921 | .99924 | .99926 | .99929 |
| 3.2 | .99931 | .99934 | .99936 | .99938 | .99940 | .99942 | .99944 | .99946 | .99948 | .99950 |
| 3.3 | .99952 | .99953 | .99955 | .99957 | .99958 | .99960 | .99961 | .99962 | .99964 | .99965 |
| 3.4 | .99966 | .99968 | .99969 | .99970 | .99971 | .99972 | .99973 | .99974 | .99975 | .99976 |
| 3.5 | .99977 | .99978 | .99978 | .99979 | .99980 | .99981 | .99981 | .99982 | .99983 | .99983 |
| 3.6 | .99984 | .99985 | .99985 | .99986 | .99986 | .99987 | .99987 | .99988 | .99988 | .99989 |
| 3.7 | .99989 | .99990 | .99990 | .99990 | .99991 | .99991 | .99992 | .99992 | .99992 | .99992 |
| 3.8 | .99993 | .99993 | .99993 | .99994 | .99994 | .99994 | .99994 | .99995 | .99995 | .99995 |
| 3.9 | .99995 | .99995 | .99996 | .99996 | .99996 | .99996 | .99996 | .99996 | .99997 | .99997 |

Table A gives the area under the curve from the left hand end to any point to the right of the mean.

**Study task**

To test your understanding of this topic attempt the following Self-test questions for Topic 4 (see next page) and then check your answers.

## Summary

The normal distribution is a family of bell-shaped symmetrical distributions which are completely specified by their mean and standard deviation. Any normal distribution can be converted to a standard normal distribution which has a mean of zero and standard deviation of one. The conversion uses the formula:

$$Z = \frac{x - \mu}{\sigma}$$

which calculates the Z-value. The Z-value may then be used to determine probabilities by referring to a table of areas under the normal curve. Alternatively, instead of calculating Z, the Excel function:

```
=NORMDIST(x,mean,standard-dev,TRUE)
```

will provide the required probability.

## Self-test questions

### I Matching

You are required to match the numbered term with the letter of the most appropriate description.

| | | | | |
|---|---|---|---|---|
| 1. | _____ | Normal distribution | A | Area between the mean and 1 standard deviation either side. |
| 2. | _____ | Z-value | B | A continuous probability distribution which is bell-shaped, symmetrical and has a single peak. |
| 3. | _____ | Bell-shaped | C | The number of standard deviations a point is from the mean of a normal distribution. |
| 4. | _____ | 0.68 | D | The shape of all normal distributions. |
| 5. | _____ | 0.50 | E | Area between the mean and 2/3 standard deviation either side. |

### II Completion statements

Complete each of the following statements with the most appropriate word or words.

1. The probability of a specific value of a continuous random variable must be ________________.
2. As the standard deviation becomes smaller the normal distribution becomes ___________.
3. As the standard deviation becomes larger the normal distribution becomes ___________.
4. For a normal distribution p(Z > 1.5) is also equal to ____________.
5. Monthly sales for catamarans average 250 boats per month with a standard deviation of 25 boats. The probability that sales will be less than 280 boats next month is __________________.

## III Exercise

The life of dry-cell flashlight batteries produced by a company is normally distributed with mean $\mu = 803$ minutes and standard deviation $\sigma_x = 41$ minutes.

a. What is the probability that a battery will last between 780 and 820 minutes?

b. What is the probability that a battery will last less than 750 minutes?

c. What is the probability that a battery will last between 700 and 900 minutes?

d. What is the probability that a battery will last more than 850 minutes?

e. The probability is 0.9 that a battery will last more than x minutes. What is x?

f. 50% of batteries last between ________ and ________ minutes. Use symmetrical limits around the mean.

g. If the company were able to reduce the standard deviation to 30 minutes without affecting the mean life, what would be the probability that a battery will last less than 750 minutes?

## Self-test solutions

### I Matching

1. B  2. C  3. D  4. A  5. E

### II Completion statements

1. zero
2. steeper or narrower
3. flatter or broader
4. 1-p(Z<1.5) or p(Z<-1.5)
5. 0.88493 Z=(280 - 250)/25 = 1.2
   p(Z≤1.2)= 0.88493

| |
|---|
| =NORMDIST(280,260,25,TRUE)=0.88493027 |

### III Exercise

a. $p(780 \le X \le 820)$
$= p(X \le 820) - p(X \le 780)$
$= p[Z \le (820 - 803)/41)] - p[Z \le (780 - 803)/41]$
$= p(Z \le 0.41) - p(Z \le -0.56)$
$= 0.65910 - (1 - 0.71226)$
$= 0.65910 - 0.28774$
$= 0.37136$

| |
|---|
| =NORMDIST(820,803,41,TRUE)-NORMDIST(780,803,41,TRUE)=0.3697666<br>Slight difference because of rounding with manual calculations. Excel result more accurate. |

b. $P(X \le 750) = p[Z \le (750 - 803)/41]$
$= p(Z \le -1.29)$
$= 1 - p(Z \le 1.29)$
$= 1 - 0.90147$
$= 0.09853$

| |
|---|
| =NORMDIST(750,803,41,TRUE)=0.09806044 |

c. $p(700 \le X \le 900)$
$= p(X \le 900) - p(X \le 700)$
$= p[Z \le (900 - 803)/41] - p[Z \le (700 - 803)/41]$
$= p(Z \le 2.37) - p(Z \le -2.51)$
$= 0.99111 - (1 - 0.99396)$
$= 0.99111 - 0.00604$
$= 0.98507$

| |
|---|
| =NORMDIST(900,803,41,TRUE)-NORMDIST(700,803,41,TRUE)=0.9850066 |

d. $p(X > 850) = 1 - p(X \le 850)$
$= 1 - p[Z \le (850 - 803)/41]$
$= 1 - p(Z \le 1.15)$
$= 1 - 0.87493$
$= 0.12507$

```
=1-NORMDIST(850,803,41,TRUE)=0.125827
```

e. If we look for 0.9 in the body of the standard normal table we find it represents a Z-value of about 1.28. Therefore X = 803 - 1.28(41) = 750.52 approximately.

Alternatively we can use the Excel function NORMSINV to find the Z score associated with 0.9 and then proceed as above:

```
=NORMSINV(0.9) returns the value 1.281552
```

f. 50% of the normal distribution lies with in the range $\mu \pm 2/3\sigma$, i.e. 803 ± 2/3(41), or between 775.7 and 830.3 minutes approximately.

g. $p(X \le 750) = p[Z \le (750 - 803/30]$
$= p(Z \le -1.77)$
$= 1 - p(Z \le 1.77)$
$= 1 - 0.961664$
$= 0.03836$

```
=NORMDIST(750,803,30,TRUE)=0.03864199
```