1. Consider the system

$$\begin{aligned}\dot{x} &= Ax + Bu = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 2 & 1 & 1 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix} u \\ y &= Cx = (1\ 0\ 0\ 0)x. \end{aligned}$$

(a) Is the system observable? .... (1p)

(b) Compute $\mathcal{V}^*$ and $\mathcal{R}^*$ contained in $\mathcal{V}^*$, and find (parameterize) ALL friends F of $\mathcal{V}^*$. .... (3p)

(c) Let F be any friend of V^*, $A_F = A + BF$, and $\Omega_F = (C^T, A_F^T C^T, \cdots, (A_F^3)^T C^T)^T$. What is the dimension of $ker\ \Omega_F$? .... (1p)

2. Consider the system

$$\begin{aligned}\dot{x} &= \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 1 \\ 1 & 1 & 0 & -2 \end{pmatrix} x + \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix} u \\ y &= (0\ 1\ 1\ 0)x, \end{aligned}$$

where $x = (x_1, x_2, x_3, x_4)^T$.

(a) Given $x(0) = (0,0,0,0)^T$, find (parameterize) all points $\bar{x}$ in $ker\ C$ that can be reached from $x(0)$ in **any** given finite time $T > 0$ with some control (i.e. $x(T) = \bar{x}$), while the trajectory of $x(t)$ is a straight line. (Here, $x(t)$ is the solution of $\dot{x} = Ax + Bu$, $x(0) = 0$.) .... (3p)

(b) If we let $x(0) = (0,1,-1,0)^T$, and $\bar{x} = (1, k, -k, 0)^T$, for what values of k $\bar{x}$ can be reached in **some** finite time from $x(0)$ while $x_4(t) = 0\ \forall t \geq 0$? .... (2p)

3. Consider

$$\begin{aligned} \dot{x} &= Ax + Bu + Ew \\ y &= Cx, \end{aligned}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \; B = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \; C = (2\ 3\ 1).$$

(a) Find the minimum constraint on E such that DDP is solvable. .... (2p)

(b) Find a $u = Fx$ that solves the DDP problem while makes the closed-loop system stable, i.e. $A + BF$ has only eigenvalues with negative real part... (2p)

(c) Verify that there exists an $E \in V^*$ such that (A, E) is controllable. Explain why even in this case the DDP problem is solvable (namely $w(t)$ will not at all influence the output). .... (1p)

4. Consider

$$\begin{aligned} \dot{x}_1 &= x_1 + x_3 + u_1 \\ \dot{x}_2 &= x_2 + x_3 - u_1 \\ \dot{x}_3 &= -x_3 + 2x_4 + u_2 \\ \dot{x}_4 &= -x_1 - x_2 + x_4 + u_1 \\ y_1 &= x_1 + x_2 \\ y_2 &= x_4 \end{aligned}$$

(a) What is the relative degree for the system? .... (1p)

(b) Convert the system into the normal form and compute the zero dynamics. (3p)

(c) When $y(t) = 0 \ \forall t \geq 0$, does it always imply $\lim_{t\to\infty} x(t) = 0$? .... (1p)